7.8.1: $x' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} x \quad |A - rI| = (3-r)(-1-r) + 4 = r^2 - 2r + 1$
$= (r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1$
$r_1 = 1: \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$r_2 = 1: \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow 2v_1 - 4v_2 = 2$
$v_1 = k \Rightarrow v_2 = \frac{2k-2}{4} = \frac{k}{2} - \frac{1}{2} \Rightarrow \underline{v} = \begin{pmatrix} k \\ \frac{1}{2}(k-1) \end{pmatrix} = k\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$
$\underline{x} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^t\right]$
Solutions $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ eigenvector go to $(-\infty, -\infty)$ in parallel lines, those below the eigenvector go to ∞ in the first quadrant,

7.8.2: $x' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} x \quad |A - rI| = (4-r)(-4-r) + 16$
$= r^2 = 0 \Rightarrow r = 0$
$r_1 = 0: \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
$r_2 = 0: \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$
$\underline{x} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}\right]$
Solutions below the eigenvector go to ∞ in the first quadrant, those above go to the third quadrant, in parallel lines

7.8.3: $x' = \begin{pmatrix} -3/2 & 1 \\ -1/4 & -1/2 \end{pmatrix} x \quad |A - rI| = (-\frac{3}{2} - r)(-\frac{1}{2} - r) + \frac{1}{4}$
$= r^2 + 2r + 1 = (r+1)^2 = 0 \Rightarrow r = -1$
$r_1 = -1: \begin{pmatrix} -1/2 & 1 \\ -1/4 & 1/2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$r_2 = -1: \begin{pmatrix} -1/2 & 1 \\ -1/4 & 1/2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$
$\underline{x} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t}\right]$
Solutions go to the origin, parallel to $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

7.8.4: $x' = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix} x \quad |A - rI| = (-3-r)(2-r) + \frac{25}{4}$
$= r^2 + r - 6 + 6\frac{1}{4} = r^2 + r + \frac{1}{4} = (r + \frac{1}{2})^2 = 0$
$r_1 = -\frac{1}{2}: \begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$r_2 = -\frac{1}{2}: \begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 0 \\ 2/5 \end{pmatrix}$
$\underline{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t/2} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} e^{-t/2}\right]$
Solutions go to the origin along the eigenvector

7.8.5: $x' = \begin{pmatrix}1&1&1\\2&1&-1\\0&-1&1\end{pmatrix}x \quad |A-rI| = (1-r)[(1-r)^2-1]$
$\qquad - (2-2r) + (-2)$

$= (1-r)(r^2-2r) + 2r - 4 = (r-2)(r-r^2+2)$

$= -(r-2)^2(r+1) = 0$

$r_1 = -1$: $\begin{pmatrix}2&1&1\\2&2&-1\\0&-1&2\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix}-3\\4\\2\end{pmatrix}$

$r_2 = 2$: $\begin{pmatrix}-1&1&1\\2&-1&-1\\0&-1&-1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$

$r_3 = 2$: $\begin{pmatrix}-1&1&1\\2&-1&-1\\0&-1&-1\end{pmatrix}\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix} \Rightarrow w = \begin{pmatrix}1\\0\\1\end{pmatrix}$

$\underline{X} = c_1\begin{pmatrix}-3\\4\\2\end{pmatrix}e^{-t} + c_2\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{2t} + c_3\left[\begin{pmatrix}0\\1\\-1\end{pmatrix}te^{2t} + \begin{pmatrix}1\\0\\1\end{pmatrix}e^{2t}\right]$

7.8.6: $x' = \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}x \quad |A-rI| = -r(r^2-1)$
$\qquad -(-r-1) + (1+r)$

$= -r^3 + r + 2r + 2 = (r+1)(-r^2+r+2) = (r+1)^2(r-2) = 0$

$r_1 = 2$: $\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$r_2 = -1$: $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix}1\\0\\-1\end{pmatrix}, \underline{w} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

$\underline{X} = c_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-t} + c_3\begin{pmatrix}1\\-1\\0\end{pmatrix}e^{-t}$

7.8.7: $x' = \begin{pmatrix}1&-4\\4&-7\end{pmatrix}x$, $x(0) = \begin{pmatrix}3\\2\end{pmatrix}$; $|A-rI| = (1-r)(-7-r)+16$

$= r^2+6r+9 = (r+3)^2 = 0$

$r_1 = -3$: $\begin{pmatrix}4&-4\\4&-4\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix}1\\1\end{pmatrix}$

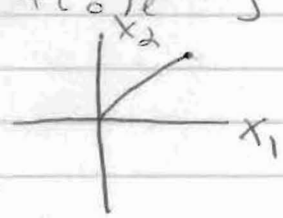
$r_2 = -3$: $\begin{pmatrix}4&-4\\4&-4\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix}\frac14+k\\k\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1/4\\0\end{pmatrix}$

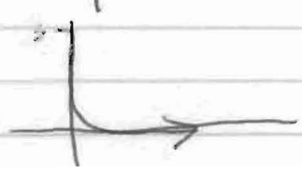
$\underline{X} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-3t} + c_2\left[\begin{pmatrix}1\\1\end{pmatrix}te^{-3t} + \begin{pmatrix}1/4\\0\end{pmatrix}e^{-3t}\right]$

$c_1 + \frac{c_2}{4} = 3$

$c_1 = 2,\ c_2 = 4$

$\underline{X}(t) = \begin{pmatrix}3+4t\\2+4t\end{pmatrix}e^{-3t}$





7.8.8: $x' = \begin{pmatrix} -5/2 & 3/2 \\ -3/2 & 1/2 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$; $|A - rI| = (-\frac{5}{2} - r)(\frac{1}{2} - r) + \frac{9}{4}$

$= r^2 + 2r + 1 = (r+1)^2 = 0$

$r_1 = -1$: $\begin{pmatrix} -3/2 & 3/2 \\ -3/2 & 3/2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

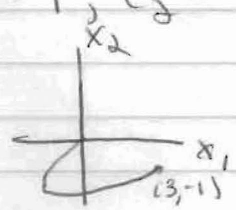
$r_2 = -1$: $\begin{pmatrix} -3/2 & 3/2 \\ -3/2 & 3/2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$

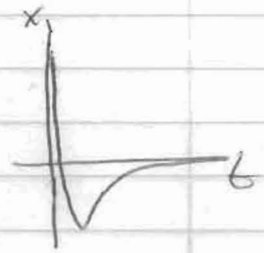
$\underline{X} = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix} e^{-t}\right)$

$c_1 - \frac{1}{3} c_2 = 3 \Rightarrow c_1 = 1,\ c_2 = -6$

$c_1 + \frac{1}{3} c_2 = -1$

$\underline{X}(t) = \begin{pmatrix} 3 - 6t \\ -1 - 6t \end{pmatrix} e^{-t}$





7.8.9: $x' = \begin{pmatrix} 2 & 3/2 \\ -3/2 & -1 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$; $|A - rI| = (2 - r)(-1 - r) + \frac{9}{4}$

$= r^2 - r + \frac{1}{4} = (r - \frac{1}{2})^2 = 0$

$r_1 = \frac{1}{2}$: $\begin{pmatrix} 3/2 & 3/2 \\ -3/2 & -3/2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

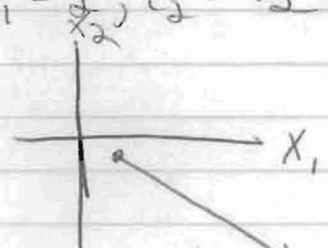
$r_2 = \frac{1}{2}$: $\begin{pmatrix} 3/2 & 3/2 \\ -3/2 & -3/2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}$

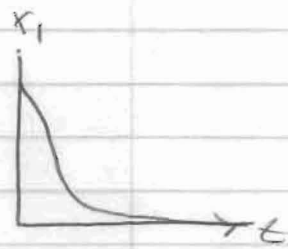
$\underline{X} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\frac{1}{2}t} + c_2\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{\frac{1}{2}t} + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix} e^{\frac{1}{2}t}\right]$

$c_1 + \frac{2}{3} c_2 = 3 \Rightarrow c_1 = 2,\ c_2 = 3/2$

$-c_1 = -2$

$\underline{X}(t) = \begin{pmatrix} 3 + \frac{3}{2}t \\ -2 - \frac{3}{2}t \end{pmatrix} e^{\frac{1}{2}t}$





7.8.10: $x' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$; $|A - rI| = (3 - r)(-3 - r) + 9$

$= r^2 = 0$

$r_1 = 0$: $\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \underline{u}$

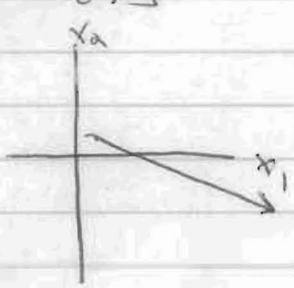
$r_2 = 0$: $\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

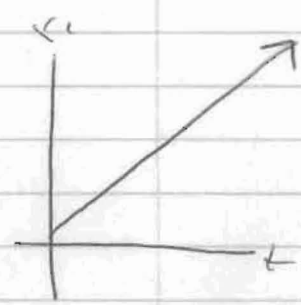
$\underline{X} = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2\left[\begin{pmatrix} 3 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$

$3c_1 + c_2 = 2$

$-c_1 = 4 \Rightarrow c_2 = 14$

$\underline{X}(t) = \begin{pmatrix} 2 + 42t \\ 4 - 14t \end{pmatrix}$





7.8.11: $x' = \begin{pmatrix}1&0&0\\-4&1&0\\3&6&2\end{pmatrix} x$; $x(0) = \begin{pmatrix}-1\\2\\-30\end{pmatrix}$ $\quad |A - rI| = (1-r)\left((1-r)(2-r)\right) = 0$

$r_1 = 2$: $\begin{pmatrix}-1&0&0\\-4&-1&0\\3&6&0\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$r_2 = 1$: $\begin{pmatrix}0&0&0\\-4&0&0\\3&6&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix}0\\1\\-6\end{pmatrix}$

$r_3 = 1$: $\begin{pmatrix}0&0&0\\-4&0&0\\3&6&1\end{pmatrix}\begin{pmatrix}\omega_1\\\omega_2\\\omega_3\end{pmatrix} = \begin{pmatrix}0\\1\\-6\end{pmatrix} \Rightarrow -4\omega_1 = 1 \Rightarrow \omega_1 = -\frac{1}{4}$

$-\frac{3}{4} + 6\omega_2 + \omega_3 = -6$

$\Rightarrow 6\omega_2 + \omega_3 = -\frac{21}{4}$

$\omega_2 = 0,\ \omega_3 = -\frac{21}{4}$

$\underline{\omega} = \begin{pmatrix}-\frac{1}{4}\\0\\-\frac{21}{4}\end{pmatrix}$

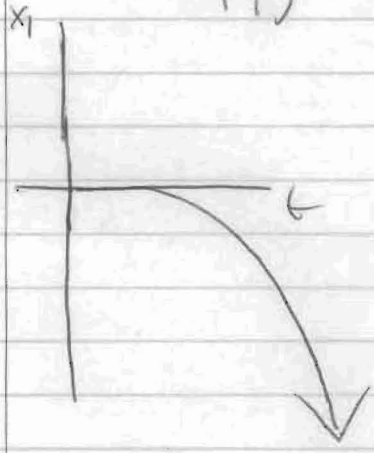
$\underline{x}(t) = c_1 e^{2t}\begin{pmatrix}0\\0\\1\end{pmatrix} + c_2 e^{t}\begin{pmatrix}0\\1\\-6\end{pmatrix} + c_3\left[\begin{pmatrix}0\\1\\-6\end{pmatrix} t e^{t} + \begin{pmatrix}-\frac{1}{4}\\0\\-\frac{21}{4}\end{pmatrix} e^{t}\right]$

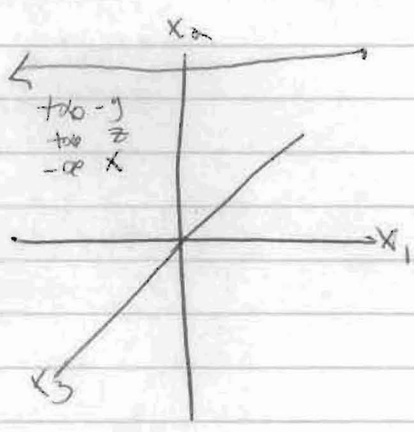
$-\frac{1}{4} c_3 = -1 \Rightarrow c_3 = 4$

$c_2 = 2 \Rightarrow c_2 = 2$

$c_1 - 6c_2 - \frac{21}{4} c_3 = -30 \Rightarrow c_1 - 12 - 21 = -30 \Rightarrow c_1 = 3$

$\underline{x}(t) = 3\begin{pmatrix}0\\0\\1\end{pmatrix} e^{2t} + \begin{pmatrix}-1\\2\\-33\end{pmatrix} e^{t} + 4\begin{pmatrix}0\\1\\-6\end{pmatrix} t e^{t}$





7.8.12; $x' = \begin{pmatrix} -5/2 & 1 & 1 \\ 1 & -5/2 & 1 \\ 1 & 1 & -5/2 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$$|A - rI| = (-5/2 - r)[(-5/2 - r)^2 - 1] - (-5/2 - r - 1) + (1 + 5/2 + r)$$
$$= (-5/2 - r)(r^2 + 5r + 21/4) + 2r + 7$$
$$= -(r^3 + 5r^2 + \tfrac{21}{4}r + \tfrac{5}{2}r^2 + \tfrac{25}{2}r + \tfrac{105}{8} - 2r - 7)$$
$$= -(r^3 + \tfrac{15}{2}r^2 + \tfrac{63}{4}r + \tfrac{49}{8}) = -(r + 7/2)(r^2 + 4r + 7/4)$$
$$= -(r + 7/2)^2 (r + 1/2)$$

$r_1 = -1/2$; $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$r_2 = -7/2$: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \underline{w} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$\underline{X} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-\frac{1}{2}t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-\frac{7}{2}t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-\frac{7}{2}t}$$

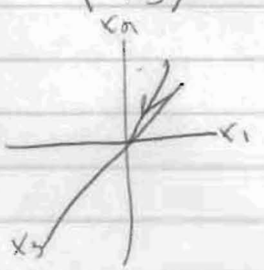
$c_1 + c_2 + c_3 = 2$

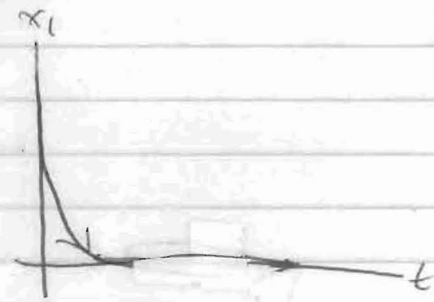
$c_1 - c_3 = 3 \Rightarrow \quad c_2 - c_3 = 4 \Rightarrow c_2 = c_3 + 4$

$c_1 - c_2 = -1 \quad\quad c_1 + 2c_3 = -2 \Rightarrow 3c_3 = -5, \; c_3 = -5/3$

$\quad\quad c_1 - c_3 = 3 \quad\quad c_1 = 4/3, \; c_2 = 7/3$

$$\underline{X}(t) = \begin{pmatrix} 4/3 \\ 4/3 \\ 4/3 \end{pmatrix} e^{-\frac{1}{2}t} + \begin{pmatrix} 2/3 \\ 5/3 \\ -7/3 \end{pmatrix} e^{-\frac{7}{2}t}$$





7.8.13: $t x' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} x \quad |A - rI| = (3-r)(-1-r) + 4$
$= r^2 - 2r + 1 = (r-1)^2 = 0$
$r_1 = 1: \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$r_2 = 1: \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\underline{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t \ln t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \right]$

7.8.14: $t x' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} x \quad |A - rI| = (1-r)(-7-r) + 16$
$= r^2 + 6r + 9 = (r+3)^2 = 0$
$r_1 = -3: \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$r_2 = -3: \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$

$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^{-3} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t^{-3} \ln t + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} t^{-3} \right]$

7.8.17: $x' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} x$

a) $|A - rI| = (1-r)[(1-r)(4-r) + 2] - (8 - 2r - 3) + (4 + 3 - 3r)$
$= (1-r)(r^2 - 5r + 6) - r + 2$.
$= (1-r)(r-2)(r-3) - (r-2) = -(r-2)[r^2 - 4r + 4]$
$= -(r-2)^3 \Rightarrow r_1 = r_2 = r_3 = 2$
$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has only one vector solution $\underline{u} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

b) $x^1(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$

c) Assume $\underline{x} = \underline{\xi} t e^{2t} + \underline{\eta} e^{2t}$

$x' = \underline{\xi} e^{2t} + 2 \underline{\xi} t e^{2t} + 2 \underline{\eta} e^{2t}$
$Ax = A \underline{\xi} t e^{2t} + A \underline{\eta} e^{2t}$
$x' = Ax \Rightarrow (\xi + 2\eta) e^{2t} + 2 \underline{\xi} t e^{2t} = A \underline{\eta} e^{2t} + A \underline{\xi} t e^{2t}$
$\Rightarrow \underline{\xi} + 2 \underline{\eta} = A \eta, \; 2 \underline{\xi} = A \underline{\xi} \Rightarrow$
$\underline{\xi} = (A - 2I) \underline{\eta}, \; 0 = (A - 2I) \underline{\xi}$

7.8.17: $(A-2I)\underline{\eta} = \underline{\xi} \Rightarrow \begin{pmatrix}-1 & 1 & 1\\ 2 & -1 & -1\\ -3 & 2 & 2\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix} = \begin{pmatrix}0\\ 1\\ -1\end{pmatrix}$

$\Rightarrow \underline{\eta} = \begin{pmatrix}1\\ 1\\ 0\end{pmatrix} \Rightarrow \underline{x}^2 = \begin{pmatrix}0\\ 1\\ -1\end{pmatrix} te^{2t} + \begin{pmatrix}1\\ 1\\ 0\end{pmatrix} e^{2t}$

d) $\underline{x} = \underline{\xi}\frac{t^2}{2}e^{2t} + \underline{\eta}\, te^{2t} + \underline{\rho}\, e^{2t}$

$\underline{x}' = \underline{\xi}\, te^{2t} + \underline{\xi}\, t^2 e^{2t} + \underline{\eta}\, e^{2t} + 2\underline{\eta}\, te^{2t} + 2\underline{\rho}\, e^{2t}$

$A\underline{x} = A\underline{\xi}\frac{t^2}{2}e^{2t} + A\underline{\eta}\, te^{2t} + A\underline{\rho}\, e^{2t}$

$\underline{x}' = A\underline{x} \Rightarrow \underline{\xi}\, t^2 e^{2t} = A\underline{\xi}\frac{t^2}{2}e^{2t} \Rightarrow 0 = (A-2I)\underline{\xi}$

$\underline{\xi}\, te^{2t} + 2\underline{\eta}\, te^{2t} = A\underline{\eta}\, te^{2t}$

$\Rightarrow \underline{\xi} = (A-2I)\underline{\eta}$

$\underline{\eta}\, e^{2t} + 2\underline{\rho}\, e^{2t} = A\underline{\rho}\, e^{2t}$

$\Rightarrow \underline{\eta} = (A-2I)\underline{\rho}$

$\begin{pmatrix}1\\ 1\\ 0\end{pmatrix} = \begin{pmatrix}-1 & 1 & 1\\ 2 & -1 & -1\\ -3 & 2 & 2\end{pmatrix}\begin{pmatrix}\rho_1\\ \rho_2\\ \rho_3\end{pmatrix} \Rightarrow \underline{\rho} = \begin{pmatrix}2\\ 0\\ 3\end{pmatrix}$

$\underline{x}^3(t) = \begin{pmatrix}0\\ 1\\ -1\end{pmatrix}\frac{t^2}{2}e^{2t} + \begin{pmatrix}1\\ 1\\ 0\end{pmatrix} te^{2t} + \begin{pmatrix}2\\ 0\\ 3\end{pmatrix} e^{2t}$

e) $\Psi(t) = \begin{bmatrix}0 & e^{2t} & te^{2t} + 2e^{2t}\\ e^{2t} & te^{2t} + e^{2t} & \frac{t^2}{2}e^{2t} + te^{2t}\\ -e^{2t} & -te^{2t} & -\frac{t^2}{2}e^{2t} + 3e^{2t}\end{bmatrix}$

f) $T = \begin{bmatrix}0 & 1 & 2\\ 1 & 1 & 0\\ -1 & 0 & 3\end{bmatrix}$ $\left[\begin{array}{ccc|ccc}0 & 1 & 2 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ -1 & 0 & 3 & 0 & 0 & 1\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1 & 2 & 2 & 1 & 1 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ 0 & 1 & 3 & 0 & 1 & 1\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1 & 2 & 2 & 1 & 1 & 0\\ 0 & -1 & -2 & -1 & 0 & 0\\ 0 & 1 & 3 & 0 & 1 & 1\end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc}1 & 0 & -2 & -1 & 1 & 0\\ 0 & 1 & 2 & 1 & 0 & 0\\ 0 & 0 & 1 & -1 & 1 & 1\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1 & 0 & 0 & -3 & 3 & 2\\ 0 & 1 & 0 & 3 & -2 & -2\\ 0 & 0 & 1 & -1 & 1 & 1\end{array}\right] \leftarrow T^{-1}$

$J = T^{-1}AT = \begin{bmatrix}-3 & 3 & 2\\ 3 & -2 & -2\\ -1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ -3 & 2 & 4\end{bmatrix}\begin{bmatrix}0 & 1 & 2\\ 1 & 1 & 0\\ -1 & 0 & 3\end{bmatrix} = \begin{bmatrix}-3 & 3 & 2\\ 3 & -2 & -2\\ -1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 2 & 5\\ 2 & 3 & 1\\ -2 & -1 & 6\end{bmatrix}$

$= \begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}$

7.8.18: $x' = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \underline{x}$

a) $|A - rI| = (5-r)[(-5-r)(3-r)+12] + 3(24-8r-16)$
$-2(24-20-4r) = (5-r)(r^2+2r-3) - 16r + 16$
$= (5-r)(r+3)(r-1) - 16(r-1) = -(r-1)[r^2-2r+1]$
$= -(r-1)^3 = 0 \Rightarrow r_1 = r_2 = r_3 = 1$

$r_1 = 1$: $\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & -3 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\underline{u}^1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\underline{u}^2 = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$

$\underline{x}^1(t) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^t$, $\underline{x}^2(t) = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} e^t$

b) $x = \xi t e^t + \eta e^t \Rightarrow x' = \underline{\xi} e^t + \underline{\xi} t e^t + \underline{\eta} e^t$
$Ax = A\xi t e^t + A\eta e^t$
$\Rightarrow \underline{\xi} t e^t = A\underline{\xi} t e^t \Rightarrow 0 = (A - I)\underline{\xi}$
$(\underline{\xi} + \underline{\eta}) e^t = A\underline{\eta} e^t \Rightarrow \underline{\xi} = (A - I)\underline{\eta}$

c)
$\underline{\xi} = c_1 \underline{u}^1 + c_2 \underline{u}^2$. A linear combination of eigenvectors is also an eigenvector
$(A - I)\underline{\eta} = \underline{\xi} = c_1 \underline{u}^1 + c_2 \underline{u}^2$
$\Rightarrow \left[\begin{array}{ccc|c} 4 & -3 & -2 & c_1 \\ 8 & -6 & -4 & 2c_2 \\ -4 & 3 & 2 & 2c_1 - 3c_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 4 & -3 & -2 & c_1 \\ 0 & 0 & 0 & 2c_2 - 2c_1 \\ 0 & 0 & 0 & 3c_1 - 3c_2 \end{array}\right]$
A solution exists only if $c_1 = c_2$

d) $c_1 = c_2 = 2 \Rightarrow \underline{\xi} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$

$\begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \Rightarrow \underline{\eta} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

$\underline{x}^3(t) = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} t e^t + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} e^t$

e) $\Psi(t) = e^t \begin{bmatrix} 1 & 0 & 2t \\ 0 & 2 & 4t \\ 2 & -3 & -2t-1 \end{bmatrix}$

7.8.18: f) $T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{bmatrix}$ $\left[\begin{matrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{matrix}\middle|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \sim \left[\begin{matrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & -1 \end{matrix}\middle|\begin{matrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ -2 & 0 & 1 \end{matrix}\right]$

$\sim \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{matrix}\middle|\begin{matrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ -2 & 3/2 & 1 \end{matrix}\right]$

$T^{-1} = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 2 & -3/2 & -1 \end{bmatrix}$

$J = T^{-1}AT = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 2 & -3/2 & -1 \end{bmatrix} \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 2 & -3/2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 4 \\ 2 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = J$

7.8.19: $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

a) $J^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$

$J^3 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}$

$J^4 = J^3 J = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^4 & 4\lambda^3 \\ 0 & \lambda^4 \end{pmatrix}$

b) $J^1 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, Assume $J^{n-1} = \begin{pmatrix} \lambda^{n-1} & (n-1)\lambda^{n-2} \\ 0 & \lambda^{n-1} \end{pmatrix}$

$J^n = JJ^{n-1} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda^{n-1} & (n-1)\lambda^{n-2} \\ 0 & \lambda^{n-1} \end{pmatrix} = \begin{pmatrix} \lambda^n & (n-1+1)\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$

Shows $J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$

c) $e^{Jt} = \sum_{i=0}^{\infty} J^i \frac{t^i}{i!} = \sum_{i=1}^{\infty} \begin{pmatrix} \lambda^i & i\lambda^{i-1} \\ 0 & \lambda^i \end{pmatrix} \frac{t^i}{i!} + I$

$= \begin{pmatrix} \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} & t\sum_{i=0}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} t^{i-1} \\ 0 & \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} \end{pmatrix} + I = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$

d) $\frac{d}{dt} e^{Jt} x_0 = \begin{pmatrix} \lambda e^{\lambda t} & e^{\lambda t} + \lambda t e^{\lambda t} \\ 0 & \lambda e^{\lambda t} \end{pmatrix} x_0$

$J e^{Jt} x_0 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} = \begin{pmatrix} \lambda e^{\lambda t} & \lambda t e^{\lambda t} + e^{\lambda t} \\ 0 & \lambda e^{\lambda t} \end{pmatrix} x_0$

$e^{Jt}(0) x_0 = I x_0 = x_0$

7.8.20: $J = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

a) $J^2 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}$

$J^3 = J^2 J = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix}$

$J^4 = J^3 J = \begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix}\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}$

b) $J^1 = \begin{pmatrix} \lambda^1 & 0 & 0 \\ 0 & \lambda^1 & 1 \\ 0 & 0 & \lambda^1 \end{pmatrix}$ Assume $J^m = \begin{pmatrix} \lambda^m & 0 & 0 \\ 0 & \lambda^m & m\lambda^{m-1} \\ 0 & 0 & \lambda^m \end{pmatrix}$

$J^{m+1} = J^m J = \begin{pmatrix} \lambda^m & 0 & 0 \\ 0 & \lambda^m & m\lambda^{m-1} \\ 0 & 0 & \lambda^m \end{pmatrix}\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^{m+1} & 0 & 0 \\ 0 & \lambda^{m+1} & (m+1)\lambda^m \\ 0 & 0 & \lambda^{m+1} \end{pmatrix}$

$\Rightarrow J^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$ by induction.

c) Determine e^{Jt} in the same way as problem 19.

d) $\Pi = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$ with $\lambda = 1$

$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} = \begin{bmatrix} e^t & 2e^t & 2te^t \\ 0 & 4e^t & 4te^t \\ 2e^t & -2e^t & -2te^t - e^t \end{bmatrix}$

The discrepancy comes from the use of $\underline{u}^2$ versus $2\underline{u}^1 + 2\underline{u}^2$ as the second column.

7.8.21: $J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

a) $J^2 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}$

$J^3 = J^2 J = \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{bmatrix}$

$J^4 = J^3 J = \begin{bmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{bmatrix}\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{bmatrix}$

$J^2 = \begin{bmatrix} \lambda^2 & 2\lambda^1 & \lambda^0 \\ 0 & \lambda^2 & 2\lambda^1 \\ 0 & 0 & \lambda^2 \end{bmatrix}$, Assume $J^{m-1} = \begin{bmatrix} \lambda^{m-1} & (m-1)\lambda^{m-2} & \frac{(m-1)(m-2)}{2}\lambda^{m-3} \\ 0 & \lambda^{m-1} & (m-1)\lambda^{m-2} \\ 0 & 0 & \lambda^{m-1} \end{bmatrix}$

$J^m = J J^{m-1} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} J^{m-1} = \begin{bmatrix} \lambda^m & (m)\lambda^{m-1} & \lambda^{m-2}\left(\frac{m^2-3m+2+2m-2}{2}\right) \\ 0 & \lambda^m & (m)\lambda^{(m-1)} \\ 0 & 0 & \lambda^m \end{bmatrix}$

By induction, the formula holds.

c) $e^{Jt} = I + \sum_{j=1}^{\infty} J^j \frac{t^j}{j!} = I + \begin{bmatrix} \sum_{j=1}^{\infty} \frac{(\lambda t)^j}{j!} & \sum_{j=1}^{\infty} j\frac{\lambda^{j-1}}{j!} t^j & \sum_{j=2}^{\infty} \frac{j(j-1)t^j\lambda^{j-2}}{2j!} \\ 0 & \sum_{j=1}^{\infty} \frac{(\lambda t)^j}{j!} & \sum_{j=1}^{\infty} \frac{j\lambda^{j-1}t^j}{j!} \\ 0 & 0 & \sum_{j=1}^{\infty} \frac{(\lambda t)^j}{j!} \end{bmatrix}$

$= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$

d) $\lambda = 2$ $\quad T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

$Te^{Jt} = \begin{bmatrix} 0 & e^{2t} & te^{2t} + 2e^{2t} \\ e^{2t} & te^{2t} + e^{2t} & \frac{t^2}{2}e^{2t} + te^{2t} \\ -e^{2t} & -te^{2t} & -\frac{t^2}{2}e^{2t} + 3e^{2t} \end{bmatrix}$