7.7.1: $x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x$, $|A - rI| = (3-r)(-2-r) + 4 = r^2 - r - 2$

$= (r-2)(r+1) = 0 \Rightarrow r_1 = 2, r_2 = -1$

$r_1 = 2: \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$r_2 = -1: \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Psi(t) = \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix}$ $\quad \Psi \underline{c} = \begin{matrix} c_1 2e^{2t} + c_2 e^{-t} \\ c_1 e^{2t} + 2c_2 e^{-t} \end{matrix}$

$\Psi(0)\underline{c} = \begin{bmatrix} 2c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 = \frac{2}{3}, c_2 = -\frac{1}{3}$

$\Psi(0)\underline{d} = \begin{bmatrix} 2d_1 + d_2 \\ d_1 + 2d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow d_1 = -\frac{1}{3}, d_2 = \frac{2}{3}$

$\underline{\Phi}(t) = \begin{bmatrix} \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} & -\frac{2}{3}e^{2t} + \frac{2}{3}e^{-t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} & -\frac{1}{3}e^{2t} + \frac{4}{3}e^{-t} \end{bmatrix}$

7.7.2: $x' = \begin{pmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{pmatrix} x$, $|A - rI| = (-\frac{3}{4} - r)(-\frac{3}{4} - r) - \frac{1}{16}$

$= r^2 + \frac{6}{4}r + \frac{1}{2} = (r+1)(r+\frac{1}{2}) = 0 \Rightarrow r_1 = -1, r_2 = -\frac{1}{2}$

$r_1 = -1: \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$r_2 = -\frac{1}{2}: \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{8} & -\frac{1}{4} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Psi(t) = \begin{pmatrix} 2e^{-t} & 2e^{-\frac{1}{2}t} \\ -e^{-t} & e^{-\frac{1}{2}t} \end{pmatrix}$ $\quad \Psi \underline{c} = \begin{matrix} c_1 2e^{-t} + c_2 2e^{-\frac{1}{2}t} \\ -c_1 e^{-t} + c_2 e^{-\frac{1}{2}t} \end{matrix}$

$\Psi(0)\underline{c} = \begin{bmatrix} 2c_1 + 2c_2 \\ -c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow c_1 = \frac{1}{4}, c_2 = \frac{1}{4}$

$\Psi(0)\underline{d} = \begin{bmatrix} 2d_1 + 2d_2 \\ -d_1 + d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow d_1 = -\frac{1}{2}, d_2 = \frac{1}{2}$

$\underline{\Phi}(t) = \begin{bmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{-\frac{1}{2}t} & -e^{-t} + e^{-\frac{1}{2}t} \\ -\frac{1}{4}e^{-t} + \frac{1}{4}e^{-\frac{1}{2}t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{-\frac{1}{2}t} \end{bmatrix}$

7.7.2: $x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x \quad |A - rI| = (2-r)(-2-r) + 3$

$= r^2 - 1 = (r+1)(r-1) = 0 \Rightarrow r = 1, -1$

$r_1 = 1: \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$r_2 = -1: \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\Psi(t) = \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{bmatrix} \quad \Psi\underline{c} = \begin{matrix} c_1 e^{t} + c_2 e^{-t} \\ c_1 e^{t} + 3c_2 e^{-t} \end{matrix}$

$\Psi(0)\underline{c} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 3c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 = 3/2, \; c_2 = -1/2$

$\Psi(0)\underline{d} = \begin{bmatrix} d_1 + d_2 \\ d_1 + 3d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow d_1 = -1/2, \; d_2 = 1/2$

$\Phi(t) = \begin{bmatrix} \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} & -\frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \\ \frac{3}{2}e^{t} - \frac{3}{2}e^{-t} & -\frac{1}{2}e^{t} + \frac{3}{2}e^{-t} \end{bmatrix}$

7.7.4: $x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x \quad |A - rI| = (1-r)(-2-r) - 4$

$= r^2 + r - 6 = (r+3)(r-2) = 0 \Rightarrow r_1 = 2, r_2 = -3$

$r_1 = 2: \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$r_2 = -3: \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\Psi(t) = \begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{bmatrix} \quad \Psi\underline{c} = \begin{matrix} c_1 e^{2t} + c_2 e^{-3t} \\ c_1 e^{2t} - 4c_2 e^{-3t} \end{matrix}$

$\Psi(0)\underline{c} = \begin{bmatrix} c_1 + c_2 \\ c_1 - 4c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 = 4/5, \; c_2 = 1/5$

$\Psi(0)\underline{d} = \begin{bmatrix} d_1 + d_2 \\ d_1 - 4d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad d_1 = 1/5, \; d_2 = -1/5$

$\Phi(t) = \begin{bmatrix} \frac{4}{5}e^{2t} + \frac{1}{5}e^{-3t} & \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t} \\ \frac{4}{5}e^{2t} - \frac{4}{5}e^{-3t} & \frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t} \end{bmatrix}$

$7.7.5: \; x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x \quad |A - rI| = (2-r)(-2-r) + 5$

$= r^2 + 1 = 0 \Rightarrow r = \pm i$

$r_1 = i: \; \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \underline{0} \Rightarrow \underline{u} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

$r_2 = -i \Rightarrow \underline{v} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$

$$\Psi(t) = \begin{bmatrix} 2\cos t + \sin t & -\cos t + 2\sin t \\ \cos t & \sin t \end{bmatrix}$$

$\Psi(0)\underline{c} = \begin{bmatrix} 2c_1 - c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 = 0, \; c_2 = -1$

$\Psi(0)\underline{d} = \begin{bmatrix} 2d_1 - d_2 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow d_1 = 1, \; d_2 = 2$

$$\Phi(t) = \begin{bmatrix} \cos t - 2\sin t & 6\sin t \\ \sin t & \cos t + 2\sin t \end{bmatrix}$$

$7.7.6: \; x' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} x \quad |A - rI| = (-1-r)(-1-r) + 4$

$= r^2 + 2r + 5 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$

$r_1 = -1 + 2i: \; \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow u = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$

$\Rightarrow r_2 = -1 - 2i, \; \underline{v} = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$

$$\Psi(t) = \begin{bmatrix} -2\sin t\, e^{-t} & 2\cos t\, e^{-t} \\ \cos t\, e^{-t} & \sin t\, e^{-t} \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} e^{-t}\cos t & -2e^{-t}\sin t \\ \frac{1}{2}e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}$$

$7.7.7: \; x' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} x \quad |A - rI| = (5-r)(1-r) + 3 = r^2 - 6r + 8$

$= (r-4)(r-2) = 0 \Rightarrow r_1 = 2, \; r_2 = 4$

$r_1 = 2: \; \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$r_2 = 4: \; \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Psi(t) = \begin{bmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{bmatrix} \quad \Psi\underline{c} = \begin{matrix} c_1 e^{2t} + c_2 e^{4t} \\ 3c_1 e^{2t} + c_2 e^{4t} \end{matrix}$

$\Psi(0)\underline{c} = \begin{bmatrix} c_1 + c_2 \\ 3c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 = -\frac{1}{2}, \; c_2 = \frac{3}{2}$

$\Psi(0)\underline{d} = \begin{bmatrix} d_1 + d_2 \\ 3d_1 + d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow d_1 = \frac{1}{2}, \; d_2 = -\frac{1}{2}$

$$\Phi(t) = \begin{bmatrix} -\frac{1}{2}e^{2t} + \frac{3}{2}e^{4t} & \frac{1}{2}e^{2t} - \frac{1}{2}e^{4t} \\ -\frac{3}{2}e^{2t} + \frac{3}{2}e^{4t} & \frac{3}{2}e^{2t} - \frac{1}{2}e^{4t} \end{bmatrix}$$

7.7.8: $x' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x \quad |A - rI| = (1-r)(-3-r)+5$

$= r^2 + 2r + 2 = 0 \Rightarrow \quad r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$r_1 = -1+i : \begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2-i \end{pmatrix} = \begin{pmatrix} 2+i \\ 5 \end{pmatrix}$

$r_2 = -1-i, \ \underline{v} = \begin{pmatrix} 2-i \\ 5 \end{pmatrix}$

$$\Psi(t) = \begin{bmatrix} 2e^{-t}\cos t - e^{-t}\sin t & e^{-t}\cos t + 2e^{-t}\sin t \\ 5e^{-t}\cos t & 5e^{-t}\sin t \end{bmatrix}$$

$$\Psi(0)\underline{c} = \begin{bmatrix} 2c_1 + c_2 \\ 5c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 = 0, \ c_2 = 1$$

$$\Psi(0)\underline{d} = \begin{bmatrix} 2d_1 + d_2 \\ 5d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow d_1 = 1/5, \ d_2 = -2/5$$

$$\Phi(t) = \begin{bmatrix} e^{-t}\cos t + 2e^{-t}\sin t & -e^{-t}\sin t \\ 5e^{-t}\sin t & e^{-t}\cos t - 2e^{-t}\sin t \end{bmatrix}$$

7.7.9: $x' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} x \quad |A - rI| = (1-r)[(1-r)(-3-r)-5]$
$- (2(-3-r) - 8) + (-10 + 8(1-r))$

$= (1-r)(r^2+2r-8) - (-2r-14) + (-8r-2)$

$= (1-r)(r+4)(r-2) - 6r + 12 = (r-2)(-r^2-3r+4-6)$

$= (r-2)(-r^2-3r-2) = -(r-2)(r+2)(r+1)$

$r_1 = 2 : \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$r_2 = -2 : \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ -8 & -5 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}$

$r_3 = -1 : \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{w} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$

$$\Psi(t) = \begin{bmatrix} 0 & -4e^{-2t} & -3e^{-t} \\ e^{2t} & 5e^{-2t} & 4e^{-t} \\ -e^{2t} & 7e^{-2t} & 2e^{-t} \end{bmatrix} \quad \Psi(0) = \begin{bmatrix} 0 & -4 & -3 \\ 1 & 5 & 4 \\ -1 & 7 & 2 \end{bmatrix}$$

$$[\underline{c}\ \underline{d}\ \underline{f}] = \Psi(0)^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{13}{12} & \frac{1}{12} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ -1 & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} -2e^{-2t} + 3e^{-t} & -e^{-2t} + e^{-t} & -e^{-2t} + e^{-t} \\ \frac{3}{2}e^{2t} + \frac{5}{2}e^{-2t} - 4e^{-t} & \frac{13}{12}e^{2t} + \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} & \frac{1}{12}e^{2t} + \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} \\ -\frac{3}{2}e^{2t} + \frac{7}{2}e^{-2t} - 2e^{-t} & -\frac{13}{12}e^{2t} + \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} & -\frac{1}{12}e^{2t} + \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} \end{bmatrix}$$

$7.7.10$: $x' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} x$ $\quad |A - rI| = (1-r)[(2-r)(-1-r)+1]$
$+ (3(-1-r)+2) + 4(3-2(2-r))$

$= (1-r)(r^2-r-1) - 3r - 1 + 8r - 4$

$= (1-r)(r^2-r-1) + 5r - 5 = (1-r)(r^2-r-1-5)$

$= (1-r)(r^2-r-6) = (1-r)(r-3)(r+2) = 0$

$r_1 = 1$: $\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

$r_2 = 3$: $\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$r_3 = -2$: $\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{w} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\Psi(t) = \begin{bmatrix} -e^{t} & e^{3t} & -e^{-2t} \\ 4e^{t} & 2e^{3t} & e^{-2t} \\ e^{t} & e^{3t} & e^{-2t} \end{bmatrix}$ $\quad \Psi(0) = \begin{bmatrix} -1 & 1 & -1 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$[\underline{c}\ \underline{d}\ \underline{f}] = \Psi(0)^{-1} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix}$

$\underline{\Phi}(t) = \begin{bmatrix} \frac{1}{6}e^{t} + \frac{1}{2}e^{3t} + \frac{1}{3}e^{-2t} & -\frac{1}{3}e^{t} + \frac{1}{3}e^{-2t} & \frac{1}{2}e^{t} + \frac{1}{2}e^{3t} - e^{-2t} \\ -\frac{2}{3}e^{t} + e^{3t} - \frac{1}{3}e^{-2t} & \frac{4}{3}e^{t} - \frac{1}{3}e^{-2t} & -2e^{t} + e^{3t} + e^{-2t} \\ -\frac{1}{6}e^{t} + \frac{1}{2}e^{3t} - \frac{1}{3}e^{-2t} & \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t} & -\frac{1}{2}e^{t} + \frac{1}{2}e^{3t} + e^{-2t} \end{bmatrix}$

$7.7.14$: $\underline{\Phi}(t) = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$

$\underline{\Phi}(s) = \begin{pmatrix} \frac{1}{2}e^{3s} + \frac{1}{2}e^{-s} & \frac{1}{4}e^{3s} - \frac{1}{4}e^{-s} \\ e^{3s} - e^{-s} & \frac{1}{2}e^{3s} + \frac{1}{2}e^{-s} \end{pmatrix}$

$\underline{\Phi}(t)\underline{\Phi}(s)_{11} = \Big(\frac{1}{4}e^{3t}e^{3s} + \frac{1}{4}e^{-t}e^{3s} + \frac{1}{4}e^{3t}e^{-s} + \frac{1}{4}e^{-t}e^{-s} + \frac{1}{4}e^{3t}e^{3s}$
$- \frac{1}{4}e^{3t}e^{-s} - \frac{1}{4}e^{3s}e^{-t} + \frac{1}{4}e^{-s}e^{-t}\Big)$

$= \frac{1}{2}e^{3(t+s)} + \frac{1}{2}e^{-(s+t)}$

$\underline{\Phi}(t)\underline{\Phi}(s)_{12} = \frac{1}{8}e^{3t}e^{3s} + \frac{1}{8}e^{3s}e^{-t} - \frac{1}{8}e^{3t}e^{-s} - \frac{1}{8}e^{-t}e^{-s}$
$+ \frac{1}{8}e^{3t}e^{3s} - \frac{1}{8}e^{3s}e^{-t} + \frac{1}{8}e^{3t}e^{-s} - \frac{1}{8}e^{-t}e^{-s} = \frac{1}{4}e^{3(t+s)} - \frac{1}{4}e^{-(t+s)}$

$\underline{\Phi}(t)\underline{\Phi}(s)_{21} = \frac{1}{2}e^{3t}e^{3s} - \frac{1}{2}e^{3s}e^{-t} + \frac{1}{2}e^{3t}e^{-s} - \frac{1}{2}e^{-t}e^{-s} + \frac{1}{2}e^{3t}e^{3s}$
$+ \frac{1}{2}e^{-t}e^{3s} - \frac{1}{2}e^{3t}e^{-s} - \frac{1}{2}e^{-t}e^{-s} = e^{3(t+s)} - e^{-(t+s)}$

$\underline{\Phi}(t)\underline{\Phi}(s)_{22} = \frac{1}{4}e^{3t}e^{3s} - \frac{1}{4}e^{3s}e^{-t} - \frac{1}{4}e^{-s}e^{3t} + \frac{1}{4}e^{-s}e^{-t} + \frac{1}{4}e^{3s}e^{3t}$
$+ \frac{1}{4}e^{-t}e^{3s} + \frac{1}{4}e^{-s}e^{3t} + \frac{1}{4}e^{-t}e^{-s} = \frac{1}{2}e^{3(s+t)} + \frac{1}{2}e^{-(t+s)}$

7.7.14: $\Phi(t)\Phi(s) = \begin{pmatrix} \frac{1}{2}e^{3(t+s)} + \frac{1}{2}e^{-(s+t)} & \frac{1}{4}e^{3(t+s)} - \frac{1}{4}e^{-(t+s)} \\ e^{3(t+s)} - e^{-(t+s)} & \frac{1}{2}e^{3(t+s)} + \frac{1}{2}e^{-(t+s)} \end{pmatrix}$

$\Phi(t+s) = \begin{pmatrix} \frac{1}{2}e^{3(t+s)} + \frac{1}{2}e^{-(t+s)} & \frac{1}{4}e^{3(t+s)} - \frac{1}{4}e^{-(t+s)} \\ e^{3(t+s)} - e^{-(t+s)} & \frac{1}{2}e^{3(t+s)} + \frac{1}{2}e^{-(t+s)} \end{pmatrix}$

$\Phi(t)\Phi(s) = \Phi(t+s)$

7.7.18: $\underline{x}' = A\underline{x}$, $\underline{x}(0) = \underline{x}^0$

a) $\underline{x} = \phi(t)$ solves the equation.

$\phi'(t) = A\phi(t) \Rightarrow \int_0^s \phi'(t)\,dt = \int_0^s A\phi(t)\,dt$

$\Rightarrow \phi(s) - \phi(0) = \int_0^s A\phi(t)\,dt$. t & s are dummy variables

so $\phi(t) - \phi(0) = \int_0^t A\phi(s)\,ds$. If ϕ is a solution, $\phi(0) = x^0$

$\Rightarrow \phi(t) = x^0 + \int_0^t A\phi(s)\,ds$

b) $\phi^0(t) = \underline{x}^0$ $\quad \phi^1(t) = x^0 + \int_0^t A\phi^0(s)\,ds$

$= x^0 + \int_0^t Ax^0\,ds$

$= x^0 + Atx^0 = (I + At)x^0$

c) $\phi^{(1)}(t) = (I + At)x^0$

Assume $\phi^{(m)}(t) = (I + At + \dots + A^m \frac{t^m}{m!})x^0$

$\phi^{(m+1)}(t) = x^0 + \int_0^t A\phi^{(m)}(s)\,ds$

$= x^0 + \int_0^t A\left(\sum_{i=0}^m A^i \frac{s^i}{i!}\right)x^0\,ds$

$= x^0\left(I + \int_0^t \sum_{i=0}^m A^{i+1}\frac{s^i}{i!}\,ds\right)$

$= x^0\left(I + \sum_{i=0}^m A^{i+1} \cdot \frac{1}{i!}\int_0^t s^i\,ds\right)$

$= x^0\left(I + \sum_{i=0}^m A^{i+1} \cdot \frac{1}{i!} \cdot \frac{t^{i+1}}{i+1}\right)$

$= x^0\left(I + \sum_{i=0}^m A^{i+1}\frac{t^{i+1}}{(i+1)!}\right) = x^0\left(\sum_{i=0}^{m+1} A^i \frac{t^i}{i!}\right)$

$= \left(I + At + \dots + A^{m+1}\frac{t^{m+1}}{(m+1)!}\right)x^0$ ✓

By induction $\phi^{(n)}(t) = (I + At + \dots + A^n \frac{t^n}{n!})x^0$

d) $\lim_{n\to\infty} \phi^{(n)}(t) = \sum_{i=0}^{\infty} A^i \frac{t^i}{i!} x_0 = e^{At}x^0$