7.5.1: $x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \underline{x}$ $\quad |A - rI| = (3-r)(-2-r)+4$
$= r^2 - r - 2 = (r-2)(r+1) = 0 \Rightarrow r_1 = 2, r_2 = -1$
$r_1 = 2$: $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow u_1 = 2u_2 \Rightarrow \underline{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$r_2 = -1$: $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2v_1 = v_2 \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
$x = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$
Solutions whose x_1-coordinate is half of their x_2 coordinate go to the origin as $t \to \infty$. The rest become unbounded. Solutions below $\underline{u}$ go to $+\infty$, those above go to $-\infty$.

7.5.2: $x' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \underline{x}$ $\quad |A - rI| = (1-r)(-4-r)+6$
$= r^2 + 3r + 2 = (r+1)(r+2) = 0 \Rightarrow r_1 = -1, r_2 = -2$
$r_1 = -1$: $\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow u_1 = u_2 \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$r_2 = -2$: $\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3v_1 = 2v_2 \Rightarrow \underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
$\underline{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$
All solutions approach the origin as $t \to \infty$.

7.5.3: $\underline{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \underline{x}$ $\quad |A - rI| = (2-r)(-2-r)+3$
$= r^2 - 1 = (r-1)(r+1) = 0 \Rightarrow r_1 = 1, r_2 = -1$
$r_1 = 1$: $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow u_1 = u_2 \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$r_2 = -1$: $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3v_1 = v_2 \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
$\underline{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$
Solutions whose x_1 coordinate is $\frac{1}{3}$ of their x_2-coordinate go to the origin as $t \to \infty$. The rest become unbounded. Solutions below $\underline{v}$ go to $+\infty$, those above go to $-\infty$.

7.5.4: $x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \underline{x}$ $\quad |A - rI| = (1-r)(-2-r)-4$
$= r^2 + r - 6 = (r+3)(r-2) = 0 \Rightarrow r_1 = 2, r_2 = -3$
$r_1 = 2$: $\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow u_1 = u_2 \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$r_2 = -3$: $\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 4v_1 = -v_2 \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$
$\underline{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$
Solutions whose x_1 coordinate is $-\frac{1}{4}$ of their x_2 coordinate go to the origin as $t \to \infty$. The rest become unbounded. Those below $\underline{v}$ go to $-\infty$, those above go to $+\infty$

7.5.5: $\underline{x}' = \begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix}\underline{x}$ $\quad |A-rI| = (-2-r)(-2-r)-1$
$= r^2+4r+3 = (r+3)(r+1) = 0 \Rightarrow r_1=-1,\ r_2=-3$
$r_1=-1$: $\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow u_1 = u_2 \Rightarrow \underline{u} = \begin{pmatrix}1\\ 1\end{pmatrix}$
$r_2=-3$: $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow v_1 = -v_2 \Rightarrow \underline{v} = \begin{pmatrix}1\\ -1\end{pmatrix}$
$\underline{x} = c_1\begin{pmatrix}1\\ 1\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\ -1\end{pmatrix}e^{-3t}$
All solutions go to the origin as $t\to\infty$

7.5.6: $\underline{x}' = \begin{pmatrix}5/4 & 3/4\\ 3/4 & 5/4\end{pmatrix}\underline{x}$ $\quad |A-rI| = (\frac{5}{4}-r)(\frac{5}{4}-r)-\frac{9}{16}$
$= r^2-\frac{10}{4}r+1 = r^2-\frac{5}{2}r+1 = (r-2)(r-\frac{1}{2}) = 0$
$r_1=\frac{1}{2}$: $\begin{pmatrix}3/4 & 3/4\\ 3/4 & 3/4\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow u_1 = -u_2 \Rightarrow \underline{u} = \begin{pmatrix}1\\ -1\end{pmatrix}$
$r_2=2$: $\begin{pmatrix}-3/4 & 3/4\\ 3/4 & -3/4\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow v_1 = v_2 \Rightarrow \underline{v} = \begin{pmatrix}1\\ 1\end{pmatrix}$
$\underline{x} = c_1\begin{pmatrix}1\\ -1\end{pmatrix}e^{\frac{1}{2}t} + c_2\begin{pmatrix}1\\ 1\end{pmatrix}e^{2t}$
Solutions not on the first eigenvector go to ∞ in the first & third quadrants as $t\to\infty$

7.5.7: $\underline{x}' = \begin{pmatrix}4 & -3\\ 8 & -6\end{pmatrix}\underline{x}$ $\quad |A-rI| = (4-r)(-6-r)+24$
$= r^2+2r = r(r+2) \Rightarrow r_1=0,\ r_2=-2$
$r_1=0$: $\begin{pmatrix}4 & -3\\ 8 & -6\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow 4u_1 = 3u_2 \Rightarrow \underline{u} = \begin{pmatrix}3\\ 4\end{pmatrix}$
$r_2=-2$: $\begin{pmatrix}6 & -3\\ 8 & -4\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow 2v_1 = v_2 \Rightarrow \underline{v} = \begin{pmatrix}1\\ 2\end{pmatrix}$
$\underline{x} = c_1\begin{pmatrix}3\\ 4\end{pmatrix} + c_2\begin{pmatrix}1\\ 2\end{pmatrix}e^{-2t}$

7.5.8: $\underline{x}' = \begin{pmatrix}3 & 6\\ -1 & -2\end{pmatrix}\underline{x}$ $\quad |A-rI| = (3-r)(-2-r)+6$
$= r^2-r = r(r-1) = 0 \Rightarrow r = 0, 1$
$r_1=0$: $\begin{pmatrix}3 & 6\\ -1 & -2\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow u_1 = -2u_2 \Rightarrow \underline{u} = \begin{pmatrix}2\\ -1\end{pmatrix}$
$r_2=1$: $\begin{pmatrix}2 & 6\\ -1 & -3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow v_1 = -3v_2 \Rightarrow \underline{v} = \begin{pmatrix}3\\ -1\end{pmatrix}$
$\underline{x} = c_1\begin{pmatrix}2\\ -1\end{pmatrix} + c_2\begin{pmatrix}3\\ -1\end{pmatrix}e^{t}$

7.5.11: $\underline{x}' = \begin{pmatrix}1 & 1 & 2\\ 1 & 2 & 1\\ 2 & 1 & 1\end{pmatrix}\underline{x}$ $\quad |A-rI| = (1-r)(\underset{(2-r)(1-r)}{\wedge}-1) - ((1-r)-2) + 2(1-4+2r)$
$= (1-r)(r^2-3r+1) + 1 + r - 6 + 4r = -r^3+3r^2-r+r^2-3r+1+5r-5$
$= -r^3+4r^2+r-4 = (r-4)(-r^2+1) = (r-4)(1+r)(1-r) = 0$
$\Rightarrow r_1 = 1,\ r_2 = 4,\ r_3 = -1$

7.5.11: $r_1 = 1$: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$r_2 = 4$: $\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$r_3 = -1$: $\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{w} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\underline{x} = c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \end{pmatrix} e^{4t} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}$

7.5.12: $\underline{x}' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \underline{x}$ $\quad |A - rI| = (3-r)((-r)(3-r)-4)$

$-2(6-2r-8) + 4(4+4r)$

$= (3-r)(r^2-3r-4) + 2(2+2r) + 16(r+1)$

$= (3-r)(r+1)(r-4) + 4(r+1) + 16(r+1) = (r+1)(20-12+7r-r^2)$

$= (r+1)(-r^2+7r+8) = -(r+1)(r-8)(r+1) \Rightarrow r_1 = -1, r_2 = -1, r_3 = 8$

$r_{1,2} = -1$: $\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \underline{u} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$

$r_3 = 8$: $\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \underline{w} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$\underline{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t}$

7.5.13: $\underline{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\quad |A - rI| = (1-r)[(1-r)(-3-r) - 5]$

$-(-6-2r-8) + (-10+8-8r)$

$= (1-r)[r^2+2r-8] - (-2r-14) - 2 - 8r$

$= (1-r)(r+4)(r-2) + 14 - 2 + 2r - 8r = (1-r)(r+4)(r-2) + 12 - 6r$

$= (r-2)[-r^2-3r+4-6] = (2-r)(r^2+3r+2) = (2-r)(r+2)(r+1)$

$\Rightarrow r_1 = 2, r_2 = -1, r_3 = -2$

$r_1 = 2$: $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$r_2 = -1$: $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$

$r_3 = -2$: $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ -8 & -5 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{w} = \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}$

7.5.13: $\underline{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} e^{-2t}$

7.5.14: $\underline{x} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \underline{x}$ $\quad |A - rI| = (1-r)[(2-r)(-1-r)+1] + (-3-3r+2) + 4(3-4+2r)$

$= (1-r)(r^2-r-1) + 8r - 4 - 3r - 1 = (1-r)(r^2-r-1) + 5r - 5$

$= (1-r)(r^2-r-1-5) = (1-r)(r-3)(r+2) \Rightarrow r_1 = 1, r_2 = -2, r_3 = 3$

$r_1 = 1$: $\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

$r_2 = -2$: $\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$r_3 = 3$: $\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{w} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\underline{x} = c_1 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}$

7.5.15: $\underline{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \underline{x}$ $\quad |A - rI| = (5-r)(1-r) + 3 = r^2 - 6r + 8$

$\Rightarrow r_1 = +2, r_2 = 4$

$r_1 = 2$: $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$r_2 = 4$: $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$ $\quad x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow c_1 = -\frac{3}{2}$ $\quad c_2 = \frac{7}{2}$

$\underline{x} = \begin{pmatrix} -\frac{3}{2} e^{2t} + \frac{7}{2} e^{4t} \\ -\frac{9}{2} e^{2t} + \frac{7}{2} e^{4t} \end{pmatrix}$

The solution moves towards ∞ in the first quadrant as $t \to \infty$

7.5.16: $\underline{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \underline{x}$ $\quad |A - rI| = (-2-r)(4-r) + 5$

$= r^2 - 2r - 3 = 0 \Rightarrow r = 3, -1$

$r_1 = 3$: $\begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

$r_2 = -1$: $\begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{x} = c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$, $x(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$

$\underline{x} = \begin{pmatrix} \frac{1}{2} e^{3t} + \frac{1}{2} e^{-t} \\ \frac{5}{2} e^{3t} + \frac{1}{2} e^{-t} \end{pmatrix}$

The solution moves towards ∞ in the first quadrant as $t \to \infty$, paralleling the eigenvector

7.5.17: $\underline{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \underline{x}$ $\quad |A - rI| = (1-\lambda)[(2-\lambda)(3-\lambda)-2] - (2-2(2-\lambda))$

$= (1-\lambda)(\lambda^2 - 5\lambda + 4) - (-2+2\lambda) = (1-\lambda)(\lambda^2 - 5\lambda + 6)$

$= (1-\lambda)(\lambda-3)(\lambda-2) = 0$

$\lambda_1 = 1: \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

$\lambda_2 = 2: \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_3 = 3: \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{w} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$\underline{x} = c_1 e^{t}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{3t}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}; \quad x(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow c_2 + 2c_3 = 2, \; 2c_1 + c_2 + 2c_3 = 0, \; -c_1 + c_3 = 1$

$c_2 = 2 - 2c_3, \; c_1 = c_3 - 1, \; 2c_3 - 2 + 2 - 2c_3 + 2c_3 = 0$

$\Rightarrow 2c_3 = 0 \Rightarrow c_3 = 0, \; c_2 = 2, \; c_1 = -1$

$x(t) = \begin{pmatrix} 2e^{2t} \\ -2e^{t} + 2e^{2t} \\ e^{t} \end{pmatrix}$

The solution goes to ∞ in the first octant as $t \to \infty$

7.5.29: $ay'' + by' + cy = 0$

a) $x_1 = y, \; x_2 = y', \; x_2' = y'' \Rightarrow \quad x_1' = x_2$
$\qquad x_2' = -\frac{b}{a}x_2 - \frac{c}{a}x_1$

$x' = Ax$ where $A = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -b/a \end{pmatrix}$

b) $|A - rI| = 0 \Leftrightarrow (-r)(-b/a - r) + c/a = 0$

$\Leftrightarrow r^2 + b/a\, r + c/a = 0 \Leftrightarrow ar^2 + br + c = 0$

7.5.30: $\underline{x}' = \begin{pmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix} \underline{x}$, $\underline{x}(0) = \begin{pmatrix} -17 \\ -21 \end{pmatrix}$

a)

$|A - rI| = (-\frac{1}{10} - r)(-\frac{1}{5} - r) - \frac{3}{400} = r^2 + \frac{3}{10} r + \frac{1}{50} - \frac{3}{400}$

$= r^2 + \frac{3}{10} r + \frac{1}{80} = 0 \Rightarrow r = \dfrac{-\frac{3}{10} \pm \sqrt{\frac{9}{100} - \frac{4}{80}}}{2} = \dfrac{-\frac{3}{10} \pm \frac{2}{10}}{2}$

$r_1 = -\frac{1}{20}$: $\begin{pmatrix} -\frac{1}{20} & \frac{3}{40} \\ \frac{1}{10} & -\frac{3}{20} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

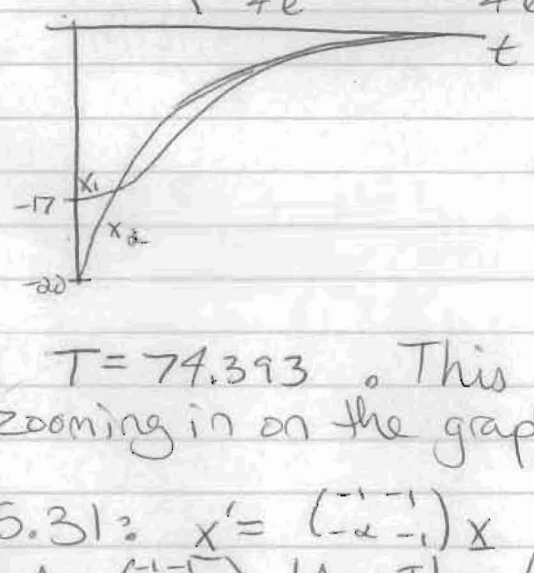
$r_2 = -\frac{1}{4}$: $\begin{pmatrix} \frac{3}{20} & \frac{3}{40} \\ \frac{1}{10} & \frac{1}{20} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\underline{x}(t) = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/4}$ $\quad 3c_1 + c_2 = -17$, $2c_1 - 2c_2 = -21$

$\Rightarrow c_1 = \frac{-55}{8}$, $c_2 = \frac{29}{8}$

$\underline{x}(t) = \begin{pmatrix} \frac{-165}{8} e^{-t/20} + \frac{29}{8} e^{-t/4} \\ \frac{-55}{4} e^{-t/20} - \frac{29}{4} e^{-t/4} \end{pmatrix}$

b)



c) $T = 74.393$. This can be found by zooming in on the graph generated in b).

7.5.31: $\underline{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \underline{x}$

a) $A = \begin{pmatrix} -1 & -1 \\ -\frac{1}{2} & -1 \end{pmatrix}$ $|A - rI| = (-1-r)^2 - \frac{1}{2} = r^2 + 2r + \frac{1}{2} = 0$

$r = \dfrac{-2 \pm \sqrt{4-2}}{2} = -1 \pm \frac{1}{\sqrt{2}}$. The equilibrium is a stable node

$r_1 = -1 + \frac{1}{\sqrt{2}}$: $\begin{pmatrix} -\frac{1}{\sqrt{2}} & -1 \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$

$r_2 = -1 - \frac{1}{\sqrt{2}}$: $\begin{pmatrix} \frac{1}{\sqrt{2}} & -1 \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$

$\underline{x} = c_1 \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{(-1+\frac{1}{\sqrt{2}})t} + c_2 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{(-1-\frac{1}{\sqrt{2}})t}$

7.5.31: b) $A = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$ $|A - rI| = (-1-r)^2 - 2 = r^2 + 2r - 1 = 0$

$r = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$

$r_1 = -1 + \sqrt{2}$: $\begin{pmatrix} -\sqrt{2} & -1 \\ -2 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}$

$r_2 = -1 - \sqrt{2}$: $\begin{pmatrix} \sqrt{2} & -1 \\ -2 & \sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} \sqrt{2} \\ 2 \end{pmatrix}$

$\underline{x} = c_1 \begin{pmatrix} -\sqrt{2} \\ 2 \end{pmatrix} e^{(-1+\sqrt{2})t} + c_2 \begin{pmatrix} \sqrt{2} \\ 2 \end{pmatrix} e^{(-1-\sqrt{2})t}$

This is a saddle.

c) $A = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix}$ $|A - rI| = (-1-r)^2 - \alpha = r^2 + 2r + 1 - \alpha = 0$

$r = \frac{-2 \pm \sqrt{4 - 4(1-\alpha)}}{2} = \frac{-2 \pm \sqrt{4 - 4 + 4\alpha}}{2} = -1 \pm \sqrt{\alpha}$

When $\sqrt{\alpha} = 1 \Rightarrow \alpha = 1$ we transition between one positive and one negative eigenvalue and two negative eigenvalues