Applied Math 9

Handout 6 on Markov Chains: The Last Word on the Invariant Distribution

Suppose that P is an $m \times m$ one step transition matrix. If $\mathbf{v}$, a row vector, is a probability distribution on the points $\{1, 2, \ldots, m\}$, and if it satisfies $\mathbf{v} = \mathbf{v}P$, then $\mathbf{v}$ is called an invariant distribution. It can be shown that for every finite state Markov chain there is at least one invariant distribution. We can solve for $\mathbf{v}$ by computing the (left) eigenvectors of P in Matlab, or by hand by using Gaussian elimination to solve the equations

$$\mathbf{v} = \mathbf{v}P, \qquad \sum_{i=1}^{m} v_i = 1.$$

Example. We solve a 3 dimensional case by hand. Let

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 0 & 2/3 \\ 1 & 0 & 0 \end{pmatrix}.$$

Writing out the system of equations, we get

$$\begin{aligned} v_1 &= \frac{1}{2}v_1 + \frac{1}{3}v_2 + v_3 \\ v_2 &= \frac{1}{2}v_1 \\ v_3 &= \frac{2}{3}v_2 \\ 1 &= v_1 + v_2 + v_3 \end{aligned}$$

which we rewrite as

$$\begin{aligned} -\frac{1}{2}v_1 + \frac{1}{3}v_2 + v_3 &= 0 \\ \frac{1}{2}v_1 - v_2 &= 0 \\ \frac{2}{3}v_2 - v_3 &= 0 \\ v_1 + v_2 + v_3 &= 1. \end{aligned}$$

The first three equations are really just two, since minus the first is just the sum of the second and third. Thus we solve

$$\begin{aligned} \frac{1}{2}v_1 - v_2 + 0v_3 &= 0 \\ 0v_1 + \frac{2}{3}v_2 - v_3 &= 0 \\ v_1 + v_2 + v_3 &= 1. \end{aligned}$$

We add the second to the third to get

$$\begin{aligned} \frac{1}{2}v_1 - v_2 + 0v_3 &= 0 \\ 0v_1 + \frac{2}{3}v_2 - v_3 &= 0 \\ v_1 + \frac{5}{3}v_2 + 0v_3 &= 1. \end{aligned}$$

We next subtract twice the first from the third to get

$$\begin{aligned} \frac{1}{2}v_1 - v_2 + 0v_3 &= 0 \\ 0v_1 + \frac{2}{3}v_2 - v_3 &= 0 \\ 0v_1 + \frac{11}{3}v_2 + 0v_3 &= 1. \end{aligned}$$

Thus $v_2 = \frac{3}{11}$, $v_3 = \frac{2}{3}v_2 = \frac{2}{11}$, $v_1 = 2v_2 = \frac{6}{11}$, and $\mathbf{v} = \frac{1}{11}(6,3,2)$. One can check that this a left eigenvector with eigenvalue 1:

$$(6,3,2)\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 0 & 2/3 \\ 1 & 0 & 0 \end{pmatrix} = (6,3,2).$$

We go back to the general discussion of the meaning of the invariant distribution. For an arbitrary Markov chain with state space $\{1,2,\ldots m\}$, denote the n-step transition probabilities by $p_{ij}^{(n)}$. We will assume the so-called *comunication condition*: for every pair of states i and j, there is some positive integer n such that $p_{ij}^{(n)} > 0$. This means that we can eventually get from i to j with positive probability. In this case the invariant distribution $\mathbf{v}$ is unique, with the following very important properties:

- *Large time probability distribution.* Suppose we are given any initial distribution $\mathbf{w}$. Then $\mathbf{w}P^n \to \mathbf{v}$ as $n \to \infty$. Put another way, every row of P^n converges to $\mathbf{v}$ as $n \to \infty$.

- *Convergence of the observed relative frequencies.* Let X_n be a Markov chain with one step transition matrix P. Let $f_1^N = \#$ of times $X_n = 1$ for $n \in \{1, \ldots, N\}$, $f_2^N = \#$ of times $X_n = 2$ for $n \in \{1, \ldots, N\}$, etc. Then with probability one,

  $$\frac{1}{N}(f_1^N, f_2^N, \ldots, f_m^N) \to \mathbf{v}$$

  as $N \to \infty$. This is a kind of law of large numbers.