APPLIED MATH 9

Handout 4 Markov Chains: Gaussian Elimination (examples)

Examples

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 5 & 0 \\ -2 & -1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The solution for a triangular matrix is very simple, we use either a forward substitution (the lower triangular matrix here) or a backward substitution (upper triangular matrix) to solve it. From first equation we can have x_1, substitute to the second equation, we obtain x_2, so on and so forth, we obtain all the components of $\mathbf{x}$. Observe we have a unique solution here because none of the diagonal elements are zero. The determinant of a triangular matrix is the multiplication of all its diagonal elements, so determinant is not zero. If we change the right hand side to zero:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 5 & 0 \\ -2 & -1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can tell that there is only one unique solution of $\mathbf{x} = \mathbf{0}$ from both way: Forward substitution or determinant is not zero. Lets look at another example by changing the second diagonal element to zero:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & -1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Now the determinant is zero, we know we either have infinite many solution or have no solution. From the first row, we have $x_1 = 1/2$ which conflicts with the second row of $x_1 = 0$, thus we have now solution. Now lets change right hand side to zero:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & -1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We know we always have $\mathbf{x} = \mathbf{0}$ and the determinant is zero, we will have other solutions. From the first two row we have $x_1 = 0$, from the third row we have

$$-x_2 + 3x_3 = 0.$$

So the solution vector can be expressed as

$$\mathbf{x} = t \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

where t is any scalar number. This is a vector pointing to direction (013) with arbitrary magnitude.

Now we see that a triangular matrix is very easy to solve. For a general matrix, we will try to transform it to a triangular matrix, this process is known as Gausian elemination:

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 5 & 0 \\ -2 & -1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

We add one third of the third row to the first row:

$$\begin{pmatrix} 4/3 & 2/3 & 0 \\ 1 & 5 & 0 \\ -2 & -1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2/3 \\ 0 \\ -1 \end{pmatrix}$$

Then we multiply $(1/5) * 2/3$ to second row and substract it from first row:

$$\begin{pmatrix} 6/5 & 0 & 0 \\ 1 & 5 & 0 \\ -2 & -1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2/3 \\ 0 \\ -1 \end{pmatrix}$$

Now we can use forward substitution solve it as

$$\mathbf{x} = t \begin{pmatrix} 5/9 \\ -1/9 \\ 0 \end{pmatrix}$$