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ON Λ-CONVERGENCE ALMOST EVERYWHERE OF MULTIPLE TRIGONOMETRIC FOURIER SERIES[1]

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Abstract: We consider one type of convergence of multiple trigonometric Fourier series intermediate between the convergence over cubes and the λ-convergence for $\lambda > 1$. The well-known result on the almost everywhere convergence over cubes of Fourier series of functions from the class $L(\ln^+ L)^d \ln^+ \ln^+ \ln^+ L([0, 2\pi)^d)$ has been generalized to the case of the Λ-convergence for some sequences Λ.

Key words: Trigonometric Fourier series, Rectangular partial sums, Convergence almost everywhere.

Suppose that d is a natural number, $\mathbb{T}^d = [-\pi, \pi)^d$ is a d-dimensional torus, and $\varphi\colon\ [0, +\infty) \to [0, +\infty)$ is a nondecreasing function. Let $\varphi(L)(\mathbb{T}^d)$ be the set of all Lebesgue measurable real-valued functions f on the torus $\mathbb{T}^d$ such that

$$\int\limits_{\mathbb{T}^d} \varphi(|f(\mathbf{t})|)d\mathbf{t} < \infty.$$

Let $f \in L(\mathbb{T}^d)$, $\mathbf{k} = (k^1, k^2, \ldots, k^d) \in \mathbb{Z}^d$, $\mathbf{x} = (x^1, x^2, \ldots, x^d) \in \mathbb{R}^d$, and $\mathbf{kx} = k^1x^1 + k^2x^2 + \ldots + k^dx^d$. Denote by

$$c_{\mathbf{k}} = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{T}^d} f(\mathbf{t})e^{-i\mathbf{kt}}\, d\mathbf{t}$$

the $\mathbf{k}$th Fourier coefficient of the function f and by

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} e^{i\mathbf{kx}} \tag{1}$$

the multiple trigonometric Fourier series of the function f.

Let $\mathbf{n} = (n^1, n^2, \ldots, n^d)$ be a vector with nonnegative integer coordinates, and let $S_{\mathbf{n}}(f, \mathbf{x})$ be the $\mathbf{n}$th rectangular partial sum of series (1):

$$S_{\mathbf{n}}(f, \mathbf{x}) = \sum_{\mathbf{k}=(k^1,\ldots,k^d):\ |k^j|\le n^j,\, 1\le j\le d} c_{\mathbf{k}} e^{i\mathbf{kx}}.$$

Denote by $\mathrm{mes}E$ the Lebesgue measure of a set E and let $\ln^+ u = \ln(u + e)$, $u \ge 0$.

In 1915, in the case $d = 1$, N.N. Luzin (see [1]) suggested that the trigonometric Fourier series of any function from $L^2(\mathbb{T})$ converges almost everywhere. A.N. Kolmogorov [2] constructed an example of a function $F \in L(\mathbb{T})$ whose trigonometric series diverges almost everywhere and, later on [3], of a function from $L(\mathbb{T})$ with the Fourier series divergent everywhere on $\mathbb{T}$. L. Carleson [4] proved that Luzin's conjecture is true: if $f \in L^2(\mathbb{T})$, then the Fourier series of the function f converges almost

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everywhere. R. Hunt [5] generalized the statement about the almost everywhere convergence of the Fourier series to the class $L(\ln^+ L)^2(\mathbb{T})$, particularly, to $L^p(\mathbb{T})$ with $p > 1$. P. Sjölin [6] generalized it to the wider class $L(\ln^+ L)(\ln^+ \ln^+ L)(\mathbb{T})$. In [7], the author showed that the condition $f \in L(\ln^+ L)(\ln^+ \ln^+ \ln^+ L)(\mathbb{T})$ is also sufficient for the almost everywhere convergence of the Fourier series of the function f. At present, the best negative result in this direction belongs to S.V. Konyagin [8]: if a function $\varphi(u)$ satisfies the condition $\varphi(u) = o(u\sqrt{\ln u/\ln\ln u})$ as $u \to +\infty$, then, in the class $\varphi(L)(\mathbb{T})$, there exists a function with the Fourier series divergent everywhere on $\mathbb{T}$.

Let us now consider the case $d \geq 2$, i.e., the case of multiple Fourier series. Let $\lambda \geq 1$. A multiple Fourier series of a function f is called λ-convergent at a point $\mathbf{x} \in \mathbb{T}^d$ if there exists a limit

$$\lim_{\min\{n^j:1\leq j\leq d\}\to+\infty} S_{\mathbf{n}}(f,\mathbf{x})$$

considered only for vectors $\mathbf{n} = (n^1, n^2, \ldots, n^d)$ such that $1/\lambda \leq n^i/n^j \leq \lambda, \quad 1 \leq i,j \leq d$. The λ-convergence is called the convergence over cubes (the convergence over squares for $d = 2$) in the case $\lambda = 1$ and the Pringsheim convergence in the case $\lambda = +\infty$, i. e., in the case without any restrictions on the relation between coordinates of vectors $\mathbf{n}$.

N.R. Tevzadze [9] proved that, if $f \in L^2(\mathbb{T}^2)$, then the Fourier series of the function f converges over cubes almost everywhere. Ch. Fefferman [10] generalized this result to functions from $L^p(\mathbb{T}^d)$, $p > 1$, $d \geq 2$. P. Sjölin [11] showed that, if a function f is from the class $L(\ln^+ L)^d(\ln^+ \ln^+ L)(\mathbb{T}^d)$, $d \geq 2$, then its Fourier series converges over cubes almost everywhere. The author [12] (see also [13]) proved the almost everywhere convergence over cubes of Fourier series of functions from the class $L(\ln^+ L)^d(\ln^+ \ln^+ \ln^+ L)(\mathbb{T}^d)$. The best current result concerning the divergence over cubes on a set of positive measure of multiple Fourier series of functions from $\varphi(L)(\mathbb{T}^d)$, $d \geq 2$, belongs to S.V. Konyagin [14]: for any function $\varphi(u) = o(u(\ln u)^{d-1}\ln\ln u)$ as $u \to +\infty$, there exists a function $F \in \varphi(L)(\mathbb{T}^d)$ with the Fourier series divergent over cubes everywhere.

On the other hand, Ch. Fefferman [15] constructed an example of a continuous function of two variables, i. e., a function from $C(\mathbb{T}^2)$ whose Fourier series diverges in the Pringsheim sense everywhere on $\mathbb{T}^2$. M. Bakhbukh and E.M. Nikishin [16] proved that there exists $F \in C(\mathbb{T}^2)$ such that its modulus of continuity satisfies the condition $\omega(F,\delta) = O\left(\ln^{-1}(1/\delta)\right)$ as $\delta \to +0$ and its Fourier series diverges in the Pringsheim sense almost everywhere. A.N. Bakhvalov [17] established that, for $m \in \mathbb{N}$ and any $\lambda > 1$, there is a function $F \in C(\mathbb{T}^{2m})$ such that the Fourier series of F is λ-divergent everywhere and the modulus of continuity of F satisfies the condition

$$\omega(F,\delta) = O\left(\ln^{-m}(1/\delta)\right), \quad \delta \to +0. \tag{2}$$

Later on, Bakhvalov [18] proved the existence of a function $F \in C(\mathbb{T}^{2m})$ satisfying condition (2) and such that its Fourier series is λ-divergent for all $\lambda > 1$ simultaneously.

Let $\Lambda = \{\lambda_\nu\}_{\nu=1}^{\infty}$ be a nonincreasing sequence of positive numbers. Assume that

$$\Omega_\Lambda = \left\{\mathbf{n} = (n^1, n^2, \ldots, n^d) \in \mathbb{N}^d : \frac{1}{1+\lambda_{n^i}} \leq \frac{n^i}{n^j} \leq 1 + \lambda_{n^j}, \quad 1 \leq i,j \leq d\right\}.$$

We will say that a multiple Fourier series of a function $f \in L(\mathbb{T}^d)$ is Λ-convergent at a point $\mathbf{x} \in \mathbb{T}^d$ if there exists a limit

$$\lim_{\mathbf{n}\in\Omega_\Lambda,\, \min\{n^j:1\leq j\leq d\}\to\infty} S_{\mathbf{n}}(f,\mathbf{x}).$$

Let us note that, if $\lambda_\nu \equiv \lambda - 1$ for some $\lambda > 1$, then the condition of Λ-convergence turns into the condition of λ-convergence defined above. And if $\lambda_\nu \to 0$ as $\nu \to \infty$, then the condition of Λ-convergence is weaker than the condition of λ-convergence for any $\lambda > 1$.

The author proved [19] that, if a sequence $\Lambda = \{\lambda_\nu\}_{\nu=1}^{\infty}$ satisfies the condition $\ln^2 \lambda_\nu = o(\ln \nu)$ as $\nu \to \infty$, then there exists a function $F \in C(\mathbb{T}^2)$ such that its Fourier series is Λ-divergent almost everywhere on $\mathbb{T}^2$.

In the present paper, we obtain the following statement that strengthens the result of [12].

Theorem 1. *Assume that a nonincreasing sequence of positive numbers* $\Lambda = \{\lambda_\nu\}_{\nu=1}^{\infty}$ *satisfies the condition*

$$\lambda_\nu = O\left(\frac{1}{\nu}\right) \tag{3}$$

and a function $\varphi\colon\ [0, +\infty) \to [0, +\infty)$ *is convex on* $[0, +\infty)$ *and such that* $\varphi(0) = 0$, $\varphi(u)u^{-1}$ *increases on* $[u_0, +\infty)$, *and* $\varphi(u)u^{-1-\delta}$ *decreases on* $[u_0, +\infty)$ *for some* $u_0 \geq 0$ *and any* $\delta > 0$. *Assume that the trigonometric Fourier series of any function* $g \in \varphi(L)(\mathbb{T})$ *converges almost everywhere on* $\mathbb{T}$. *Then, for any* $d \geq 2$, *the Fourier series of any function* f *from the class* $\varphi(L)(\ln^+ L)^{d-1}(\mathbb{T}^d)$ *is* Λ*-convergent almost everywhere on* $\mathbb{T}^d$.

Theorem 1 and the result of paper [7] imply the following statement.

Theorem 2. *Let a nonincreasing sequence of positive numbers* $\Lambda = \{\lambda_\nu\}_{\nu=1}^{\infty}$ *satisfy condition* (3), $d \geq 2$. *Then the Fourier series of any function* f *from the class*

$$L(\ln^+ L)^d(\ln^+ \ln^+ \ln^+ L)(\mathbb{T}^d)$$

is Λ*-convergent almost everywhere on* $\mathbb{T}^d$.

P r o o f of Theorem 1. Let a sequence $\Lambda = \{\lambda_\nu\}_{\nu=1}^{\infty}$ and a function φ satisfy the conditions of the theorem. Let $\varphi_d(u) = \varphi(u)(\ln^+ u)^{d-1}$ for short. Without loss of generality, we can consider only functions φ_d such that the functions $\varphi_d(\sqrt{u})$ are concave on $[0, +\infty)$. Otherwise, we can consider the functions $\varphi_d(u + a_d) - b_d$ (with appropriate constants a_d and b_d) instead of φ_d. The corresponding class $\varphi_d(L)(\mathbb{T}^d)$ will be the same in this case.

Denote by $S_n(f, \mathbf{x})$ the nth cubic partial sum of the Fourier series of the function f:

$$S_n(f, \mathbf{x}) = S_{\mathbf{n}}(f, \mathbf{x}), \quad \text{where} \quad \mathbf{n} = (n, \ldots, n).$$

Suppose that

$$M(f, \mathbf{x}) = \sup_{n \in \mathbb{N}} |S_n(f, \mathbf{x})|,$$

$$M_\Lambda(f, \mathbf{x}) = \sup_{\mathbf{n} \in \Omega_\Lambda} |S_{\mathbf{n}}(f, \mathbf{x})|.$$

Under the conditions of the theorem (see [12, formula (3.1) and Lemma 3]), there are constants $K_d > 0$ and $y_d \geq 0$ such that

$$\operatorname{mes}\Big\{\mathbf{x} \in \mathbb{T}^d : M(f, \mathbf{x}) > y\Big\} \leq \frac{K_d}{y}\bigg(\int\limits_{\mathbb{T}^d} \varphi_d(|f(\mathbf{x})|)\, d\mathbf{x} + 1\bigg), \quad y > y_d, \quad f \in \varphi_d(L)(\mathbb{T}^d). \tag{4}$$

Using (4), we will prove that, for every $y > y_d$ and $f \in \varphi_d(L)(\mathbb{T}^d)$,

$$\operatorname{mes}\Big\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(f, \mathbf{x}) > y\Big\} \leq \frac{A_d}{y}\bigg(\int\limits_{\mathbb{T}^d} \varphi_d(|f(\mathbf{x})|)\, d\mathbf{x} + 1\bigg) \tag{5}$$

and, for every $f \in \varphi_{d+1}(L)(\mathbb{T}^d)$,

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f,\mathbf{x})d\mathbf{x} \le B_d\Bigg(\int\limits_{\mathbb{T}^d} \varphi_{d+1}(|f(\mathbf{x})|)\, d\mathbf{x} + 1\Bigg), \tag{6}$$

where A_d is independent of f and y; B_d is independent of f.

The proof is by induction on d. Consider the base case, i. e., $d = 1$: statement (5) immediately follows from (4) because $M(f,\mathbf{x}) = M_\Lambda(f,\mathbf{x})$ in the one-dimensional case. Similarly, (6) is a consequence of [5, Theorem 2].

Let $d \ge 2$. Suppose that statements (5) and (6) hold for $d-1$ and let us show that the same is true for d.

First, let us prove the validity of (5). Let $\mathbf{n} = (n^1, n^2, \dots, n^d) \in \Omega_\Lambda$. According to (3), there is an absolute constant $C > 0$ such that $\lambda_\nu \nu \le C$ for all natural numbers ν. Combining this with the definition of Ω_Λ, we obtain that, for all $i, j \in \{1, 2, \dots, d\}$,

$$|n^i - n^j| \le C. \tag{7}$$

Recall that, if $\mathbf{n} = (n^1, n^2, \dots, n^d)$, then the following representation holds for the $\mathbf{n}$th rectangular partial sum of the Fourier series of the function f:

$$S_{\mathbf{n}}(f,\mathbf{x}) = \frac{1}{\pi^d} \int\limits_{\mathbb{T}^d} \prod_{j=1}^{d} D_{n^j}(t^j) f(x^1 + t^1, \dots, x^d + t^d)\, dt^1 \dots dt^d, \tag{8}$$

where $D_n(t) = \sin((n + 1/2)t)/(2\sin(t/2))$ is the one-dimensional Dirichlet kernel of order n. Let us add to and subtract from the d-dimensional Dirichlet kernel $\prod\limits_{j=1}^{d} D_{n^j}(t^j)$ of order $\mathbf{n}$ the sum

$$\sum_{k=2}^{d} \left(\prod_{j=1}^{k} D_{n^1}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) \right)$$

(here and in what follows, we suppose that all products $\prod$ with an upper index less than a lower one are equal to 1). Rearranging the terms, we obtain

$$\prod_{j=1}^{d} D_{n^j}(t^j) = \sum_{k=1}^{d-1} \left(\prod_{j=1}^{k} D_{n^1}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) - \prod_{j=1}^{k+1} D_{n^1}(t^j) \prod_{j=k+2}^{d} D_{n^j}(t^j) \right) + \prod_{j=1}^{d} D_{n^1}(t^j) =$$

$$= \sum_{k=2}^{d} \left(\prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) \left(D_{n^k}(t^k) - D_{n^1}(t^k) \right) \right) + \prod_{j=1}^{d} D_{n^1}(t^j).$$

From this and (8), it follows that

$$S_{\mathbf{n}}(f,\mathbf{x})=\sum_{k=2}^{d}\frac{1}{\pi^d}\int\limits_{\mathbb{T}^d}\left(\prod_{j=1}^{k-1}D_{n^1}(t^j)\prod_{j=k+1}^{d}D_{n^j}(t^j)\left(D_{n^k}(t^k)-D_{n^1}(t^k)\right)\right)\times$$

$$\times f(x^1+t^1,\dots,x^d+t^d)\,dt^1\dots dt^d+\frac{1}{\pi^d}\int\limits_{\mathbb{T}^d}\prod_{j=1}^{d}D_{n^1}(t^j)f(x^1+t^1,\dots,x^d+t^d)\,dt^1\dots dt^d=$$

$$=\sum_{k=2}^{d}\frac{1}{\pi^d}\int\limits_{\mathbb{T}}\left(D_{n^k}(t^k)-D_{n^1}(t^k)\right)\times$$

$$\times\left(\int\limits_{\mathbb{T}^{d-1}}\prod_{j=1}^{k-1}D_{n^1}(t^j)\prod_{j=k+1}^{d}D_{n^j}(t^j)f(x^1+t^1,\dots,x^d+t^d)\,dt^1\dots dt^{k-1}dt^{k+1}\dots dt^d\right)dt^k+S_{n^1}(f,\mathbf{x}). \tag{9}$$

Note that the latter term on the right hand side of (9) is the n^1th cubic partial sum of the Fourier series of the function f. By (7), for all $k\in\{2,3,\dots,d\}$ and $t\in\mathbb{T}$, we have $|D_{n^k}(t)-D_{n^1}(t)|\le C$. Combining this with (9), we obtain

$$|S_{\mathbf{n}}(f,\mathbf{x})|\le\sum_{k=2}^{d}\frac{C}{\pi^d}\int\limits_{\mathbb{T}}\Bigg|\int\limits_{\mathbb{T}^{d-1}}\prod_{j=1}^{k-1}D_{n^1}(t^j)\prod_{j=k+1}^{d}D_{n^j}(t^j)\times$$

$$\times f(x^1+t^1,\dots,x^{k-1}+t^{k-1},t^k,x^{k+1}+t^{k+1},\dots,x^d+t^d)\,dt^1\dots dt^{k-1}dt^{k+1}\dots dt^d\Bigg|\,dt^k+|S_{n^1}(f,\mathbf{x})|.$$

Applying the definitions of $M_\Lambda(f,\mathbf{x})$ and $M(f,\mathbf{x})$, from the latter estimate, we obtain

$$M_\Lambda(f,\mathbf{x})\le M(f,\mathbf{x})+\frac{C}{\pi}\sum_{k=2}^{d}\int\limits_{\mathbb{T}}\sup_{\mathbf{n}=(n^1,n^2,\cdots,n^d)\in\Omega_\Lambda}\Bigg|\frac{1}{\pi^{d-1}}\int\limits_{\mathbb{T}^{d-1}}\prod_{j=1}^{k-1}D_{n^1}(t^j)\prod_{j=k+1}^{d}D_{n^j}(t^j)\times$$

$$\times f(x^1+t^1,\dots,x^{k-1}+t^{k-1},t^k,x^{k+1}+t^{k+1},\dots,x^d+t^d)\,dt^1\dots dt^{k-1}dt^{k+1}\dots dt^d\Bigg|\,dt^k= \tag{10}$$

$$=M(f,\mathbf{x})+\frac{C}{\pi}\sum_{k=2}^{d}M_k(f,\mathbf{x}),$$

where $M_k(f,\mathbf{x})$ denotes the kth term of the sum on the left hand side of the equality in (10). Let $k\in\{2,3,\dots,d\}$. Consider $M_k(f,\mathbf{x})$. Denote by g_{k,t^k} the function of $d-1$ variables that can be obtained from the function f by fixing the kth variable t^k:

$$g_{k,t^k}(t^1,\dots,t^{k-1},t^{k+1},\dots,t^d)=f(t^1,\dots,t^{k-1},t^k,t^{k+1},\dots,t^d),\quad(t^1,\dots,t^{k-1},t^{k+1},\dots,t^d)\in\mathbb{T}^{d-1}.$$

Define Ω'_Λ as the set of $\mathbf{m}_k=(m^1,\dots,m^{k-1},m^{k+1},\dots,m^d)\in\mathbb{N}^{d-1}$ such that $\mathbf{m}=(m^1,\dots,m^d)\in\Omega_\Lambda$. Note that, in view of the invariance of Ω_Λ with respect to a rearrangement of variables, the set Ω'_Λ is independent of k. Suppose that $\mathbf{n}'_k=(n^1,\dots,n^1,n^{k+1},\dots,n^d)\in\mathbb{N}^{d-1}$. Then

$$\frac{1}{\pi^{d-1}}\int\limits_{\mathbb{T}^{d-1}}\prod_{j=1}^{k-1}D_{n^1}(t^j)\prod_{j=k+1}^{d}D_{n^j}(t^j)\times$$

$$\times f(x^1+t^1,\dots,x^{k-1}+t^{k-1},t^k,x^{k+1}+t^{k+1},\dots,x^d+t^d)\,dt^1\dots dt^{k-1}dt^{k+1}\dots dt^d=$$

$$= S_{\mathbf{n}'_k}\left(g_{k,t^k}, (x^1,\dots,x^{k-1},x^{k+1},\dots,x^d)\right)$$

and

$$M_k(f,\mathbf{x}) = \int\limits_{\mathbb{T}} \sup_{\mathbf{n}'_k\in\Omega'_\Lambda} \left|S_{\mathbf{n}'_k}\left(g_{k,x^k}, (x^1,\dots,x^{k-1},x^{k+1},\dots,x^d)\right)\right| dx^k.$$

Further,

$$\operatorname{mes}\left\{\mathbf{x}\in\mathbb{T}^d : M_k(f,\mathbf{x})>y\right\} = 2\pi\operatorname{mes}\left\{(x^1,\dots,x^{k-1},x^{k+1},\dots,x^d)\in\mathbb{T}^{d-1} : M_k(f,\mathbf{x})>y\right\} \le$$
$$\le \frac{2\pi}{y}\int\limits_{\mathbb{T}^{d-1}} M_k(f,\mathbf{x})\,dx^1\dots dx^{k-1}dx^{k+1}\dots dx^d =$$
$$= \frac{2\pi}{y}\int\limits_{\mathbb{T}^d} \sup_{\mathbf{n}'_k\in\Omega'_\Lambda}\left|S_{\mathbf{n}'_k}\left(g_{k,x^k}, (x^1,\dots,x^{k-1},x^{k+1},\dots,x^d)\right)\right| d\mathbf{x} =$$
$$= \frac{2\pi}{y}\int\limits_{\mathbb{T}}\Bigg(\int\limits_{\mathbb{T}^{d-1}} \sup_{\mathbf{n}'_k\in\Omega'_\Lambda}\left|S_{\mathbf{n}'_k}\left(g_{k,x^k}, (x^1,\dots,x^{k-1},x^{k+1},\dots,x^d)\right)\right| dx^1\dots dx^{k-1}dx^{k+1}\dots dx^d\Bigg)\,dx^k. \tag{11}$$

From this, applying the induction hypothesis (more precisely, statement (6) for the dimension $d-1$) to the inner integral on the right hand part of (11), we obtain

$$\operatorname{mes}\left\{\mathbf{x}\in\mathbb{T}^d : M_k(f,\mathbf{x})>y\right\} \le \frac{2\pi}{y}\int\limits_{\mathbb{T}}\Big(B_{d-1}\int\limits_{\mathbb{T}^{d-1}}\varphi_d(|f(\mathbf{x})|)\,dx^1\dots dx^{k-1}dx^{k+1}\dots dx^d+1\Big)\,dx^k \le$$
$$\le \frac{(2\pi)^2B_{d-1}}{y}\Bigg(\int\limits_{\mathbb{T}^d}\varphi_d(|f(\mathbf{x})|)\,d\mathbf{x}+1\Bigg). \tag{12}$$

According to (10),

$$\left\{\mathbf{x}\in\mathbb{T}^d : M_\Lambda(f,\mathbf{x})>y\right\} \subset \left\{\mathbf{x}\in\mathbb{T}^d : M(f,\mathbf{x})>\frac{y}{2}\right\}\bigcup\Bigg(\bigcup_{k=2}^{d}\left\{\mathbf{x}\in\mathbb{T}^d : M_k(f,\mathbf{x})>\frac{\pi y}{2(d-1)C}\right\}\Bigg). \tag{13}$$

Combining (13), (4) and (12), we obtain (5) with the constant $A_d = 2K_d + 8\pi(d-1)^2B_{d-1}C$.

Now, we only need to prove the validity of statement (6). To this end, let us use statement (5) proved above.

From (5), it follows that the majorant $M_\Lambda(f,\mathbf{x})$ is finite almost everywhere on $\mathbb{T}^d$ for all $f\in\varphi_d(L)(\mathbb{T}^d)$, in particular, for all $f\in L^2(T^d)$. Applying Stein's theorem on limits of sequences of operators [20, Theorem 1], we see that the operator $M_\Lambda(f,\cdot)$ is of weak type $(2,2)$, i.e., there is a constant $A_d^2>0$ such that, for all $y>0$ and $f\in L^2(T^d)$,

$$\operatorname{mes}\left\{\mathbf{x}\in\mathbb{T}^d : M_\Lambda(f,\mathbf{x})>y\right\} \le \frac{A_d^2}{y^2}\int\limits_{\mathbb{T}^d}|f(\mathbf{x})|^2\,d\mathbf{x}. \tag{14}$$

Similarly, from [20, Theorem 3], we can obtain the following refinement of statement (5): there is a constant $\bar{A}_d>0$ such that, for all $y\ge \bar{y}_d/2=\bar{A}_d$ and $f\in\varphi_d(L)(\mathbb{T}^d)$,

$$\operatorname{mes}\left\{\mathbf{x}\in\mathbb{T}^d : M_\Lambda(f,\mathbf{x})>y\right\} \le \int\limits_{\mathbb{T}^d}\varphi_d\left(\frac{\bar{A}_d|f(\mathbf{x})|}{y}\right)\,d\mathbf{x} \le \frac{\bar{A}_d}{y}\int\limits_{\mathbb{T}^d}\varphi_d(|f(\mathbf{x})|)\,d\mathbf{x}. \tag{15}$$

Further, let $f \in \varphi_d(L)(\mathbb{T}^d)$ and $y > 0$. Suppose that

$$g(x) = g_y(x) = \begin{cases} f(x), & |f(x)| > y, \\ 0, & |f(x)| \le y; \end{cases} \qquad h(x) = h_y(x) = f(x) - g(x).$$

Define $\lambda_f(y) = \operatorname{mes}\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(f, \mathbf{x}) > y\}$. Then

$$\lambda_f(y) \le \operatorname{mes}\Big\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(g, \mathbf{x}) > y/2\Big\} + \operatorname{mes}\Big\{\mathbf{x} \in \mathbb{T}^d : M_\Lambda(h, \mathbf{x}) > y/2\Big\} = \lambda_g(y/2) + \lambda_h(y/2).$$

From this, using the equality

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x})\, d\mathbf{x} = -\int\limits_0^\infty y\, d\lambda_f(y) = \int\limits_0^\infty \lambda_f(y)\, dy$$

(see, for example, [21, Chapter 1, § 13, formula (13.6)]), we obtain

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x})\, d\mathbf{x} \le \bar{y}_d (2\pi)^d + \int\limits_{\bar{y}_d}^\infty \lambda_f(y)\, dy \le \bar{y}_d (2\pi)^d + \int\limits_{\bar{y}_d}^\infty \lambda_g\left(\frac{y}{2}\right) dy + \int\limits_{\bar{y}_d}^\infty \lambda_h\left(\frac{y}{2}\right) dy. \tag{16}$$

Taking into account that $g \in \varphi_d(L)(\mathbb{T}^d)$ and $h \in L^\infty(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$ and applying estimate (15) to $\lambda_g(y/2)$ and estimate (14) to $\lambda_h(y/2)$, from (16), we obtain

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x})\, d\mathbf{x} \le \bar{y}_d (2\pi)^d + 2\bar{A}_d \int\limits_{\bar{y}_d}^\infty \left(\frac{1}{y} \int\limits_{\mathbb{T}^d} \varphi_d(|g(\mathbf{t})|)\, d\mathbf{t}\right) dy + 4A_d^2 \int\limits_{\bar{y}_d}^\infty \left(\frac{1}{y^2} \int\limits_{\mathbb{T}^d} |h(\mathbf{t})|^2\, d\mathbf{t}\right) dy =$$

$$= \bar{y}_d (2\pi)^d + 2\bar{A}_d \int\limits_{\bar{y}_d}^\infty \left(\frac{1}{y} \int\limits_{\{\mathbf{t} \in \mathbb{T}^d:\ |f(\mathbf{t})| > y\}} \varphi_d(|f(\mathbf{t})|)\, d\mathbf{t}\right) dy + 4A_d^2 \int\limits_{\bar{y}_d}^\infty \left(\frac{1}{y^2} \int\limits_{\{\mathbf{t} \in \mathbb{T}^d:\ |f(\mathbf{t})| \le y\}} |f(\mathbf{t})|^2\, d\mathbf{t}\right) dy. \tag{17}$$

Applying Fubibi's theorem to the integrals on the right hand side of (17), we conclude that

$$\int\limits_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x})\, d\mathbf{x} \le 2\bar{A}_d \int\limits_{\{\mathbf{t} \in \mathbb{T}^d:\ |f(\mathbf{t})| > \bar{y}_d\}} \varphi_d(|f(\mathbf{t})|) \left(\int\limits_{\bar{y}_d}^{|f(\mathbf{t})|} \frac{dy}{y}\right) d\mathbf{t} +$$

$$+ 4A_d^2 \int\limits_{\mathbb{T}^d} |f(\mathbf{t})|^2 \left(\int\limits_{|f(\mathbf{t})|}^\infty \frac{dy}{y^2}\right) d\mathbf{t} + \bar{y}_d (2\pi)^d,$$

hence, statement (6) follows easily.

Finally, the Λ-convergence of the Fourier series of an arbitrary function from the class $\varphi_d(L)(\mathbb{T}^d)$ can be obtained from (5) by means of standard arguments (see, for example, [12, Lemma 3]). Theorem 1 is proved. □

REFERENCES

1. **Luzin N.N.** Integral and trigonometric series. Moscow – Leningrad: GITTL. 1951. 550 p. [in Russian]
2. **Kolmogoroff A.** Une série de Fourier – Lebesgue divergente preque partout // Fund. math., 1923. Vol. 4. P. 324–328.
3. **Kolmogoroff A.** Une série de Fourier – Lebesgue divergente partout // C. r. Acad. sci. Paris, 1926. Vol. 183. P. 1327–1329.
4. **Carleson L.** On convergence and growth of partial sums of Fourier series // Acta math., 1966. Vol. 116, no. 1–2. P. 135–157. DOI: 10.1007/BF02392815
5. **Hunt R.A.** On the convergence of Fourier series // Orthogonal expansions and their continuous analogues. SIU Press, Carbondale, Illinois, 1968. P. 235–255.
6. **Sjölin P.** An inequality of Paley and convergence a.e. of Walsh-Fourier series // Arkiv för mat., 1969. Vol. 7. P. 551–570. DOI: 10.1007/BF02590894
7. **Antonov N.Yu.** Convergence of Fourier series // East Journal on Approximations, 1996. Vol. 2, no. 2. P. 187–196.
8. **Konyagin S.V.** On everywhere divergence of trigonometric Fourier series// Sb. Math., 2000. Vol. 191, no. 1. P. 97–120. DOI: 10.1070/SM2000v191n01ABEH000449
9. **Tevzadze N.R.** The convergence of the double Fourier series of a square integrable function // Soobshch. AN GSSR, 1970. Vol. 58, no. 2. P. 277–279. [in Russian]
10. **Fefferman C.** On the convergence of multiple Fourier series // Bull. Amer. Math. Soc., 1971. Vol. 77, no. 5. P. 744–745. DOI: 10.1090/S0002-9904-1971-12793-3
11. **Sjölin P.** Convergence almost everywhere of ertain singular integrals and multiple Fourier series // Arkiv för mat., 1971. Vol. 9, no. 1. P. 65–90. DOI: 10.1007/BF02383638
12. **Antonov N.Yu.** Almost everywhere convergence over cubes of multiple trigonometric Fourier series // Izv. Math., 2004. Vol. 68, no. 2, P. 223–241. DOI: 10.1070/IM2004v068n02ABEH000472
13. **Antonov N.Yu.** On the almost everywhere convergence of sequences of multiple rectangular Fourier sums // Proc. Steklov Inst. Math., 2009. Vol. 264, Suppl. 1. P. S1–S18. DOI: 10.1134/S0081543809050010
14. **Konyagin S.V.** On divergence of trigonometric Fourier series over cubes // Acta Sci. Math. (Szeged), 1995. Vol. 61. P. 305–329.
15. **Fefferman C.** On the divergence of multiple Fourier series // Bull. Amer. Math. Soc., 1971. Vol. 77, no. 2. P. 191–195. DOI: 10.1090/S0002-9904-1971-12675-7
16. **Bakhbukh M., Nikishin E. M.** The convergence of the double Fourier series of continuous functions // Siberian Math. Journal, 1973. Vol. 14, no. 6. P. 832–839. DOI: 10.1007/BF00975888
17. **Bakhvalov A. N.** Divergence everywhere of the Fourier series of continuous functions of several variables // Sb. Math., 1997. Vol. 188, no. 8. P. 1153–1170. DOI: 10.1070/SM1997v188n08ABEH000240.
18. **Bakhvalov A. N.** λ-divergence of the Fourier series of continuous functions of several variables // Mathematical Notes, 2002. Vol. 72, no. 3–4. P. 454–465. DOI: 10.4213/mzm438
19. **Antonov N.Yu.** On divergence almost everywhere of Fourier series of continuous functions of two variables // Izvestiya of Saratov University. New ser. Ser. Math. Mech. Inform., 2014. Vol. 14, iss. 4, pt. 2. P. 495–502. [in Russian]
20. **Stein E.M.** On limits of sequences of operators // Ann. Math., 1961. Vol. 74, no. 1. P. 140–170.
21. **Zygmund A.** Trigonometric series, vol. 1. New York: Cambridge Univ. Press, 1959. 383 p.