AM 034 — Applied Mathematics - II

Brown University | **Spring 2019**

Solutions to Homework, Set 3 | **Due February 27**

3.1 (20 points) Which of the following matrices is diagonalizable ?

$$(\mathbf{a})\quad \begin{bmatrix} -731 & 228 & 690 \\ -410 & 131 & 385 \\ -650 & 202 & 614 \end{bmatrix}, \qquad (\mathbf{b})\quad \begin{bmatrix} -41 & 18 & -6 \\ -126 & 55 & -18 \\ -63 & 27 & -8 \end{bmatrix}, \qquad (\mathbf{c})\quad \begin{bmatrix} -14 & 6 & 0 \\ -24 & 10 & 3 \\ 33 & -15 & 10 \end{bmatrix}.$$

Find minimal polynomials for each of the above matrices.

Solution. *Grading:*
+5 for resolvent, +5 for minimal polynomials, +1 for checking that the minimal polynomial annihilates the matrix, +9 for determination of the matrices to be diagonalizable.

(a) The resolvent $\mathbf{R}_\lambda(\mathbf{A}) = (\lambda\,\mathbf{I} - \mathbf{A})^{-1}$ of the given matrix is

$$\mathbf{R}_\lambda = \frac{1}{(\lambda-9)(\lambda-4)(\lambda-1)} \begin{bmatrix} 2664 - 745\lambda + \lambda^2 & -612 + 228\lambda & 30(-87+23\lambda) \\ 1490 - 410\lambda & \lambda^2 + 117\lambda - 334 & 5(-293+77\lambda) \\ 2330 - 650\lambda & -538 + 202\lambda & -2281 + 600\lambda + \lambda^2 \end{bmatrix}.$$

Mathematica codes:

```
A = {{-731, 228, 690}, {-410, 131, 385}, {-650, 202, 614}}
Eigenvalues[A]
{9, 4, 1}
RA[lambda_] =
 Simplify[Inverse[
    lambda*IdentityMatrix[3] - A]*(lambda - 9)*(lambda - 4)*(lambda - 1)]
{{2664 - 745 lambda + lambda^2, -612 + 228 lambda,
  30 (-87 + 23 lambda)}, {1490 - 410 lambda, -334 + 117 lambda +
   lambda^2,
  5 (-293 + 77 lambda)}, {2330 - 650 lambda, -538 +
   202 lambda, -2281 + 600 lambda + lambda^2}}
```

Since all eigenvalues are distinct, the given matrix is diagonalizable and its minimal polynomial is

$$\psi(\lambda) = (\lambda-9)(\lambda-4)(\lambda-1) = \lambda^3 - 14\,\lambda^2 + 49\,\lambda - 36.$$

Trace of the matrix $\mathbf{A}$ is $-731+131+614 = 14 = 9+4+1$ and its determinant is $36 = 9*4*1$.

Now we check whether the characteristic polynomial is its minimal polynomial:

$$\mathbf{A}^3 - 14\,\mathbf{A}^2 + 49\,\mathbf{A} - 36\,\mathbf{I} = \mathbf{0}.$$

Mathematica confirms:

```
CharacteristicPolynomial[A, x]
36 - 49 x + 14 x^2 - x^3
MatrixPower[A, 3] - 14 MatrixPower[A, 2] + 49*A - 36*IdentityMatrix[3]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

There is another way to check the minimal polynomial with *Mathematica*:

```
(A - IdentityMatrix[3]).(A - 4*IdentityMatrix[3]).(A - 9*IdentityMatrix[3])
```

(b) Matrix **B** has three positive eigenvalues $\lambda = 1, 1, 4$ (so $\lambda = 1$ is a double eigenvalue), and its resolvent is

$$\mathbf{R}_\lambda(\mathbf{B}) = (\lambda\mathbf{I} - \mathbf{B})^{-1} = \frac{1}{\psi(\lambda)} \begin{bmatrix} \lambda - 46 & 18 & -6 \\ -126 & \lambda + 50 & -18 \\ 63 & 27 & \lambda - 13 \end{bmatrix},$$

where

$$\psi(\lambda) = (\lambda - 4)(\lambda - 1) = \lambda^2 - 5\lambda + 4$$

is the minimal polynomial for the given 3×3 matrix. We check with *Mathematica*:

```
B = {{-41, 18, -6}, {-126, 55, -18}, {-63, 27, -8}}
Eigenvalues[B]
{4, 1, 1}
Simplify[Inverse[lambda*IdentityMatrix[3] - B]]
{{(-46 + lambda)/((-4 + lambda) (-1 + lambda)), 18/(
    4 - 5 lambda + lambda^2), -(6/(4 - 5 lambda + lambda^2))}, {-(
     126/(4 - 5 lambda + lambda^2)), (50 + lambda)/(
    4 - 5 lambda + lambda^2), -(18/(4 - 5 lambda + lambda^2))}, {-(
     63/(4 - 5 lambda + lambda^2)), 27/(
    4 - 5 lambda + lambda^2), (-13 +
     lambda)/((-4 + lambda) (-1 + lambda))}
```

Mathematica shows that $\psi(\lambda)$ is the minimal polynomial for matrix **B**:

```
B.B - 5*B + 4*IdentityMatrix[3]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

(c) Consider matrix **C** and calculate its resolvent:

```
c = {{-14, 6, 0}, {-24, 10, 3}, {33, -15, 10}}
Eigenvalues[c]
{4, 1, 1}
```

Then we ask *Mathematica* to find its resolvent:

```
Simplify[Inverse[lambda*IdentityMatrix[3] - c]]
{{(145 - 20 lambda + lambda^2)/((-4 + lambda) (-1 + lambda)^2), (
  6 (-10 + lambda))/((-4 + lambda) (-1 + lambda)^2),
  18/((-4 + lambda) (-1 + lambda)^2)}, {(
  339 - 24 lambda)/((-4 + lambda) (-1 + lambda)^2), (-140 + 4 lambda +
    lambda^2)/((-4 + lambda) (-1 + lambda)^2), (
  3 (14 + lambda))/((-4 + lambda) (-1 + lambda)^2)}, {(
  30 + 33 lambda)/((-4 + lambda) (-1 + lambda)^2), -((
   3 (4 + 5 lambda))/((-4 + lambda) (-1 + lambda)^2)), (2 +
    lambda)^2/((-4 + lambda) (-1 + lambda)^2)}}
Expand[(x - 1)^2*(x - 4)]
```

So its resolvent is

$$\mathbf{R}_\lambda = (\lambda\mathbf{I} - \mathbf{C})^{-1} = \frac{1}{\psi(\lambda)} \begin{bmatrix} \lambda^2 - 20\lambda + 145 & 6\,(\lambda - 10) & 18 \\ 339 - 24\lambda & \lambda^2 + 4\lambda - 140 & 3\,(\lambda + 14) \\ 33\lambda + 30 & -3\,(5\lambda + 4) & (\lambda + 2)^2 \end{bmatrix},$$

where

$$\psi(\lambda) = (\lambda - 4)\,(\lambda - 1)^2 = \lambda^3 - 6\,\lambda^2 + 9\,\lambda - 4$$

is the minimal polynomial for the given 3×3 matrix.

We check with *Mathematica*

```
MatrixPower[c, 3] - 6 MatrixPower[c, 2] + 9*c - 4*IdentityMatrix[3]
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

but

```
(c-IdentityMatrix[3]).(c-4*IdentityMatrix[3])
{{126, -54, 18}, {315, -135, 45}, {63, -27, 9}}
```

which is not zero matrix. Therefore the characteristic polynomial for matrix **C** is its minimal polynomial, and therefore this matrix is not diagonalizable.

```
clear all; close all; clc;

% this script checks the minimal polynomials of the matrices A, B, and C
% and computes the Sylvester auxiliary matrices for A and B

A = [-731 228 690;
     -410 131 385;
     -650 202 614];
```

```

B = [-41 18 -6;
     -126 55 -18;
     -63 27 -8];

C = [-14 6 0;
     -24 10 3;
      33 -15 10];

% compute eigenvalues of A and B
LA = eig(A);
LB = eig(B);
LC = eig(C);

Id = eye(3);

% eigenvalues of A are distinct, so we just check that the characteristic
% polynomial is indeed the minimal polynomial
(A-LA(1)*Id)*(A-LA(2)*Id)*(A-LA(3)*Id)

% for B, the minimal polynomial is either the characteristic polynomial or
% the polynomial with only a single factor of each eigenvalue
(B-LB(1)*Id)*(B-LB(2)*Id)*(B-LB(3)*Id)
(B-LB(1)*Id)*(B-LB(2)*Id)

% for C, we repeat the steps from B
(C-LC(1)*Id)*(C-LC(2)*Id)*(C-LC(3)*Id)
(C-LC(1)*Id)*(C-LC(2)*Id)

% compute Sylvester matrices of A
ZA1 = (A-LA(2)*Id).*(A-LA(3)*Id)/((LA(1)-LA(2))*(LA(1)-LA(3)));
ZA2 = (A-LA(1)*Id).*(A-LA(3)*Id)/((LA(2)-LA(1))*(LA(2)-LA(3)));
ZA3 = (A-LA(1)*Id).*(A-LA(2)*Id)/((LA(3)-LA(1))*(LA(3)-LA(1)));

% compute Sylvester matrices of B
ZB1 = (B-LB(2)*Id)/(LB(1)-LB(2));
ZB2 = (B-LB(1)*Id)/(LB(2)-LB(1));
```

3.2 (20 points) For matrices **A** and **B** from parts (a) and (b) of the previous exercise, find

$$\mathbf{\Phi}_A(t) = \frac{\sin\left(\sqrt{\mathbf{A}}\,t\right)}{\sqrt{\mathbf{A}}}, \quad \mathbf{\Phi}_B(t) = \frac{\sin\left(\sqrt{\mathbf{B}}\,t\right)}{\sqrt{\mathbf{B}}}, \quad \mathbf{\Psi}_A(t) = \cos\left(\sqrt{\mathbf{A}}\,t\right), \quad \mathbf{\Psi}_B(t) = \cos\left(\sqrt{\mathbf{B}}\,t\right).$$

Solution. *Grading: PART A: +5 for Φ, +5 for Ψ. PART B: +5 for Φ, +5 for Ψ.*

(a) Using Sylvester's formula, we define the matrix-functions:

$$\begin{aligned}\mathbf{\Phi}(t) &= \left.\frac{\sin\left(\sqrt{\lambda}t\right)}{\sqrt{\lambda}}\right|_{\lambda=9}\mathbf{Z}_{(9)} + \left.\frac{\sin\left(\sqrt{\lambda}t\right)}{\sqrt{\lambda}}\right|_{\lambda=4}\mathbf{Z}_{(4)} + \left.\frac{\sin\left(\sqrt{\lambda}t\right)}{\sqrt{\lambda}}\right|_{\lambda=1}\mathbf{Z}_{(1)} \\ &= \frac{\sin 3t}{3}\,\mathbf{Z}_{(9)} + \frac{\sin 2t}{2}\,\mathbf{Z}_{(4)} + \sin t\,\mathbf{Z}_{(1)}\end{aligned}$$

and

$$\begin{aligned}\mathbf{\Psi}(t) &= \cos\left(\sqrt{\lambda}t\right)\Big|_{\lambda=9}\mathbf{Z}_{(9)} + \cos\left(\sqrt{\lambda}t\right)\Big|_{\lambda=64}\mathbf{Z}_{(4)} + \cos\left(\sqrt{\lambda}t\right)\Big|_{\lambda=64}\mathbf{Z}_{(1)} \\ &= \cos 3t\,\mathbf{Z}_{(9)} + \cos 2t\,\mathbf{Z}_{(4)} + \cos t\,\mathbf{Z}_{(1)},\end{aligned}$$

where

$$\mathbf{Z}_{(9)} = \frac{(\mathbf{A}-4\mathbf{I})\,(\mathbf{A}-\mathbf{I})}{(9-4)\,(9-1)} = \begin{bmatrix} -99 & 36 & 90 \\ -55 & 20 & 50 \\ -88 & 32 & 80 \end{bmatrix},$$

$$\mathbf{Z}_{(4)} = \frac{(\mathbf{A}-9\,\mathbf{I})\,(\mathbf{A}-\mathbf{I})}{(4-1)\,(4-9)} = \begin{bmatrix} 20 & -20 & -10 \\ 10 & -10 & -5 \\ 18 & -18 & -9 \end{bmatrix},$$

$$\mathbf{Z}_{(1)} = \frac{(\mathbf{A}-4\,\mathbf{I})\,(\mathbf{A}-9\,\mathbf{I})}{(1-4)\,(1-9)} = \begin{bmatrix} 80 & -16 & -80 \\ 45 & -9 & -45 \\ 70 & -14 & -70 \end{bmatrix}$$

are Sylvester's auxiliary matrices. They are orthogonal projections on the eigenspaces:

$$\mathbf{Z}_{(1)}^2 = \mathbf{Z}_{(1)}, \qquad \mathbf{Z}_{(4)}^2 = \mathbf{Z}_{(4)}, \qquad \mathbf{Z}_{(9)}^2 = \mathbf{Z}_{(9)},$$

and

$$\mathbf{Z}_{(1)} + \mathbf{Z}_{(4)} + \mathbf{Z}_{(9)} = \mathbf{I},$$

the identity matrix.

Mathematica codes:

```
A = {{-731, 228, 690}, {-410, 131, 385}, {-650, 202, 614}}
Z1 = Simplify[(A - 4*IdentityMatrix[3]).(A -
       9*IdentityMatrix[3])/(1 - 4)/(1 - 9)]
{{80, -16, -80}, {45, -9, -45}, {70, -14, -70}}
Z1.Z1
{{80, -16, -80}, {45, -9, -45}, {70, -14, -70}}
Z4 = Simplify[(A - IdentityMatrix[3]).(A - 9*IdentityMatrix[3])/(4 -
      1)/(4 - 9)]
{{20, -20, -10}, {10, -10, -5}, {18, -18, -9}}
```

```
Z4.Z4
{{20, -20, -10}, {10, -10, -5}, {18, -18, -9}}
Z9 = Simplify[(A - IdentityMatrix[3]).(A - 4*IdentityMatrix[3])/(9 -
      1)/(9 - 4)]
{{-99, 36, 90}, {-55, 20, 50}, {-88, 32, 80}}
Z9.Z9
{{-99, 36, 90}, {-55, 20, 50}, {-88, 32, 80}}
Z1 + Z4 + Z9
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
```

(b) Similarly, independently which square root was chosen, we have that

$$\begin{aligned}\mathbf{\Phi}(t) &= \left.\frac{\sin\left(\sqrt{\lambda}t\right)}{\sqrt{\lambda}}\right|_{\lambda=1} \mathbf{Z}_{(1)} + \left.\frac{\sin\left(\sqrt{\lambda}t\right)}{\sqrt{\lambda}}\right|_{\lambda=4} \mathbf{Z}_{(4)} \\ &= \sin t\,\mathbf{Z}_{(1)} + \frac{\sin 2t}{2}\,\mathbf{Z}_{(4)} = \sin t \begin{bmatrix} 15 & -6 & 2 \\ 42 & -17 & 6 \\ 21 & -9 & 4 \end{bmatrix} + \frac{\sin 2t}{2} \begin{bmatrix} -14 & 6 & -2 \\ -42 & 18 & -6 \\ -21 & 9 & -3 \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{\Psi}(t) &= \left.\cos\left(\sqrt{\lambda}t\right)\right|_{\lambda=1} \mathbf{Z}_{(1)} + \left.\cos\left(\sqrt{\lambda}t\right)\right|_{\lambda=4} \mathbf{Z}_{(4)} \\ &= \cos t \begin{bmatrix} 15 & -6 & 2 \\ 42 & -17 & 6 \\ 21 & -9 & 4 \end{bmatrix} . + \cos 2t \begin{bmatrix} -14 & 6 & -2 \\ -42 & 18 & -6 \\ -21 & 9 & -3 \end{bmatrix} .\end{aligned}$$

Here

$$\begin{aligned}\mathbf{Z}_{(4)} &= \frac{\mathbf{B} - \mathbf{I}}{4-1} = \begin{bmatrix} -14 & 6 & -2 \\ -42 & 18 & -6 \\ -21 & 9 & -3 \end{bmatrix}, \\ \mathbf{Z}_{(1)} &= \frac{\mathbf{B} - 4\,\mathbf{I}}{1-4} = \begin{bmatrix} 15 & -6 & 2 \\ 42 & -17 & 6 \\ 21 & -9 & 4 \end{bmatrix}\end{aligned}$$

are Sylvester's auxiliary matrices. They are orthogonal projections on the eigenspaces:

$$\mathbf{Z}_{(1)}^2 = \mathbf{Z}_{(1)}, \qquad \mathbf{Z}_{(4)}^2 = \mathbf{Z}_{(4)}, \qquad \mathbf{Z}_{(1)}\,\mathbf{Z}_{(4)} = \mathbf{0},$$

and

$$\mathbf{Z}_{(1)} + \mathbf{Z}_{(4)} = \mathbf{I},$$

the identity matrix.

```
B = {{-41, 18, -6}, {-126, 55, -18}, {-63, 27, -8}}
Z4 = (B - IdentityMatrix[3])/(4 - 1)
Z1 = (B - 4*IdentityMatrix[3])/(1 - 4)
Z1.Z1 - Z1
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
Z4.Z4 - Z4
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
Z1.Z4
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

3.3 (20 points) For each matrix from the previous exercise, show that the matrix-functions $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ satisfy the matrix differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathbf{\Phi}_A(t) + \mathbf{A}\mathbf{\Phi}(t) = \mathbf{0} \qquad \text{or} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathbf{\Psi}_A(t) + \mathbf{A}\,\mathbf{\Psi}(t) = \mathbf{0}.$$

(Similar equations are valid for matrix $\mathbf{B}$.) For each matrix $\mathbf{A}$ and $\mathbf{B}$, what initial conditions do these matrix-functions $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ satisfy?

Solution. *Grading: PART A: +5 for showing for Φ, +5 for showing for Ψ. PART B: +5 for showing for Φ, +5 for showing for Ψ.*

(a) First, we check the initial conditions:

$$\mathbf{\Phi}(0) = \frac{\sin(3*0)}{3}\mathbf{Z}_{(9)} + \frac{\sin(2*0)}{2}\mathbf{Z}_{(4)} + \frac{\sin(1*0)}{1}\mathbf{Z}_{(1)} = \mathbf{0},$$
$$\mathbf{\Psi}(0) = \cos(3*0)\,\mathbf{Z}_{(9)} + \cos(2*0)\,\mathbf{Z}_{(4)} + \cos(1*0)\,\mathbf{Z}_{(1)} = \mathbf{Z}_{(9)} + \mathbf{Z}_{(4)} + \mathbf{Z}_{(1)} = \mathbf{I}.$$

For the first derivatives, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{\Phi}(t) = \dot{\mathbf{\Phi}}(t) = \cos 3t\,\mathbf{Z}_{(9)} + \cos 2t\,\mathbf{Z}_{(4)} + \cos t\,\mathbf{Z}_{(1)},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{\Psi}(t) = \dot{\mathbf{\Psi}}(t) = -3\,\sin 3t\,\mathbf{Z}_{(9)} - 2\,\sin 2t\,\mathbf{Z}_{(4)} - \sin t\,\mathbf{Z}_{(1)}.$$

Setting $t = 0$, we get

$$\dot{\mathbf{\Phi}}(0) = \mathbf{Z}_{(9)} + \mathbf{Z}_{(4)} + \mathbf{Z}_{(1)} = \mathbf{I}.$$

Obviously, $\dot{\mathbf{\Psi}}(0) = \mathbf{0}$. Calculating second derivatives, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathbf{\Phi}(t) = -3\,\sin 3t\,\mathbf{Z}_{(9)} - 2\,\sin 2t\,\mathbf{Z}_{(4)} - \sin t\,\mathbf{Z}_{(1)},$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathbf{\Psi}(t) = -9\,\cos 3t\,\mathbf{Z}_{(9)} - 4\,\cos 2t\,\mathbf{Z}_{(4)} - \cos t\,\mathbf{Z}_{(1)}.$$

To prove that matrix functions $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ satisfy the second order matrix differential equation, we have to show that

$$9\,\mathbf{Z}_{(9)} = \mathbf{A}\,\mathbf{Z}_{(9)} \qquad 4\,\mathbf{Z}_{(4)} = \mathbf{A}\,\mathbf{Z}_{(4)} \qquad \text{and} \qquad \mathbf{Z}_{(1)} = \mathbf{A}\,\mathbf{Z}_{(1)}.$$

Indeed, it is easy to check with a software.

$$\mathbf{A}\,\mathbf{Z}_{(9)} = \begin{bmatrix} -891 & 324 & 810 \\ -495 & 180 & 450 \\ -792 & 288 & 720 \end{bmatrix} = 9 \begin{bmatrix} -99 & 36 & 90 \\ -55 & 20 & 50 \\ -88 & 32 & 80 \end{bmatrix} = 9\,\mathbf{Z}_{(9)},$$

$$\mathbf{A}\,\mathbf{Z}_{(4)} = \begin{bmatrix} 80 & -80 & -40 \\ 40 & -40 & -20 \\ 72 & -72 & -36 \end{bmatrix} = 4 \begin{bmatrix} 20 & -20 & -10 \\ 10 & -10 & -5 \\ 18 & -18 & -9 \end{bmatrix} = 4\,\mathbf{Z}_{(4)},$$

$$\mathbf{A}\,\mathbf{Z}_{(1)} = \mathbf{Z}_{(1)} = \begin{bmatrix} 80 & -16 & -80 \\ 45 & -9 & -45 \\ 70 & -14 & -70 \end{bmatrix}.$$

Mathematica codes:

```
A = {{-731, 228, 690}, {-410, 131, 385}, {-650, 202, 614}}
Z9.A/9
A.Z9/9
{{-99, 36, 90}, {-55, 20, 50}, {-88, 32, 80}}
A.Z4/4
{{20, -20, -10}, {10, -10, -5}, {18, -18, -9}}
Z1.A
{{80, -16, -80}, {45, -9, -45}, {70, -14, -70}}
```

(b) Calculating derivatives, we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathbf{\Phi}(t) = -\sin t \begin{bmatrix} 15 & -6 & 2 \\ 42 & -17 & 6 \\ 21 & -9 & 4 \end{bmatrix} - 2\,\sin 2t \begin{bmatrix} -14 & 6 & -2 \\ -42 & 18 & -6 \\ -21 & 9 & -3 \end{bmatrix},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathbf{\Psi}(t) = -\cos t \begin{bmatrix} 15 & -6 & 2 \\ 42 & -17 & 6 \\ 21 & -9 & 4 \end{bmatrix} - 4\,\cos 2t \begin{bmatrix} -14 & 6 & -2 \\ -42 & 18 & -6 \\ -21 & 9 & -3 \end{bmatrix}.$$

Next step is to evaluate products of matrices:

$$\mathbf{B}\,\mathbf{Z}_{(1)} = \begin{bmatrix} -41 & 18 & -6 \\ -126 & 55 & -18 \\ -63 & 27 & -8 \end{bmatrix} \begin{bmatrix} 15 & -6 & 2 \\ 42 & -17 & 6 \\ 21 & -9 & 4 \end{bmatrix} = \begin{bmatrix} 15 & -6 & 2 \\ 42 & -17 & 6 \\ 21 & -9 & 4 \end{bmatrix} = \mathbf{Z}_{(1)},$$

$$\mathbf{B}\,\mathbf{Z}_{(4)} = \begin{bmatrix} -41 & 18 & -6 \\ -126 & 55 & -18 \\ -63 & 27 & -8 \end{bmatrix} = \begin{bmatrix} -56 & 24 & -8 \\ -168 & 72 & -24 \\ -84 & 36 & -12 \end{bmatrix} = 4\,\mathbf{Z}_{(4)}.$$

Therefore, both matrix-functions, $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$, satisfy the second order differential equation. The initial conditions follow from the following equations:

$$\begin{aligned}
&\mathbf{\Phi}(0) = \mathbf{0}, \\
&\mathbf{\Psi}(0) = \mathbf{Z}_{(1)} + \mathbf{Z}_{(4)} = \begin{bmatrix} 15 & -6 & 2 \\ 42 & -17 & 6 \\ 21 & -9 & 4 \end{bmatrix} + \begin{bmatrix} -14 & 6 & -2 \\ -42 & 18 & -6 \\ -21 & 9 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}, \\
&\dot{\mathbf{\Phi}}(0) = \mathbf{Z}_{(1)} + \mathbf{Z}_{(4)} = \mathbf{I}, \\
&\dot{\mathbf{\Psi}}(0) = \mathbf{0}.
\end{aligned}$$

3.4 (20 points) For the matrices $\mathbf{A} = \begin{bmatrix} -731 & 228 & 690 \\ -410 & 131 & 385 \\ -650 & 202 & 614 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -41 & 18 & -6 \\ -126 & 55 & -18 \\ -63 & 27 & -8 \end{bmatrix}$, construct the exponential matrices

$$\mathbf{U}_A(t) = e^{\mathbf{A}\,t} \qquad \text{and} \qquad \mathbf{U}_B(t) = e^{\mathbf{B}\,t}.$$

Show that these matrix-functions are (unique) solutions of the following matrix initial value problems:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{U}_A(t) = \mathbf{A}\,\mathbf{U}_A(t), \quad \mathbf{U}_A(0) = \mathbf{I} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{U}_B(t) = \mathbf{B}\,\mathbf{U}_B(t), \quad \mathbf{U}_B(0) = \mathbf{I},$$

respectively.

Solution. *Grading: +5 for constructing $\mathbf{U}_A(t)$; +5 for constructing $\mathbf{U}_B(t)$; +5 for showing that $\mathbf{U}_A(t)$ satisfies the differential equation; +5 for showing that $\mathbf{U}_B(t)$ satisfies its differential equation.*

(a) Since Sylvester's auxiliary matrices are known, we build the exponential matrix

$$\begin{aligned}
e^{\mathbf{A}\,t} &= \mathbf{U}_A(t) = e^{9t}\,\mathbf{Z}_{(9)} + e^{4t}\,\mathbf{Z}_{(4)} + e^{t}\mathbf{Z}_{(1)} \\
&= e^{9t}\begin{bmatrix} -99 & 36 & 90 \\ -55 & 20 & 50 \\ -88 & 32 & 80 \end{bmatrix} + e^{4t}\begin{bmatrix} 20 & -20 & -10 \\ 10 & -10 & -5 \\ 18 & -18 & -9 \end{bmatrix} + e^{t}\begin{bmatrix} 80 & -36 & -80 \\ 45 & -9 & -45 \\ 70 & -14 & -70 \end{bmatrix}.
\end{aligned}$$

This matrix function is a (unique) solution of the matrix initial value problem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{U}_A(t) = \mathbf{A}\,\mathbf{U}_A(t), \quad \mathbf{U}_A(0) = \mathbf{I}.$$

To check that matrix function $\mathbf{U}(t)$ is a solution to the above differential equation, we differentiate each term:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{U}_A(t) = 9\,e^{9t}\,\mathbf{Z}_{(9)} + 4\,e^{4t}\,\mathbf{Z}_{(4)} + e^{t}\,\mathbf{Z}_{(1)}.$$

Since we showed previously that

$$9\,\mathbf{Z}_{(9)} = \mathbf{A}\,\mathbf{Z}_{(9)}, \quad 4\,\mathbf{Z}_{(4)} = \mathbf{A}\,\mathbf{Z}_{(4)}, \quad \mathbf{Z}_{(1)} = \mathbf{A}\,\mathbf{Z}_{(1)},$$

we conclude that the matrix function $\mathbf{U}_A(t)$ is a solution of the given matrix differential equation. The initial condition follows from

$$\mathbf{Z}_{(9)} + \mathbf{Z}_{(4)} + \mathbf{Z}_{(1)} = \mathbf{I}.$$

The matrix $\mathbf{B}$ is diagonalizable because its minimal polynomial is a product of simple terms $\psi(\lambda) = (\lambda - 4)\,(\lambda - 1)$. Sylvester's auxiliary matrices become

$$\mathbf{Z}_{(4)} = \frac{\mathbf{B} - \mathbf{I}}{4-1} = \begin{bmatrix} -14 & 6 & -2 \\ -42 & 18 & -6 \\ -21 & 9 & -3 \end{bmatrix},$$

$$\mathbf{Z}_{(1)} = \frac{\mathbf{B} - 4\,\mathbf{I}}{1-4} = \begin{bmatrix} 15 & -6 & 2 \\ 42 & -17 & 6 \\ 21 & -9 & 4 \end{bmatrix}$$

where $\mathbf{I}$ is the identity matrix.

Now we construct the exponential and cosine (3×3) matrices:

$$\mathbf{U}(t) = e^{\mathbf{B}\,t} = e^{t}\,\mathbf{Z}_{(1)} + e^{4t}\,\mathbf{Z}_{(4)} = e^{t}\begin{bmatrix} 15 & -6 & 2 \\ 42 & -17 & 6 \\ 21 & -9 & 4 \end{bmatrix} + e^{4t}\begin{bmatrix} -14 & 6 & -2 \\ -42 & 18 & -6 \\ -21 & 9 & -3 \end{bmatrix}.$$

To show that the matrix-function $\mathbf{U}(t) = e^{\mathbf{B}\,t}$ is a solution of the differential equation $\dot{\mathbf{y}} = \mathbf{B}\,\mathbf{y}$, we first differentiate it

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{U}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\,e^{\mathbf{B}\,t} = e^{t}\begin{bmatrix} 15 & -6 & 2 \\ 42 & -17 & 6 \\ 21 & -9 & 4 \end{bmatrix} + 4\,e^{4t}\begin{bmatrix} -14 & 6 & -2 \\ -42 & 18 & -6 \\ -21 & 9 & -3 \end{bmatrix},$$

and then calculate the products: $\mathbf{B}\,\mathbf{Z}_{(1)}$ and $\mathbf{B}\,\mathbf{Z}_{(4)}$ that was done in part (3).

3.5 (20 points) Find square roots of the matrix $\mathbf{C} = \begin{bmatrix} -14 & 6 & 0 \\ -24 & 10 & 3 \\ 33 & -15 & 10 \end{bmatrix}$.

Solution. *Grading: +5 for constructing the resolvent. +5 for evaluating residues. +5 for final constructing of all four square roots. +5 for showing that the square of the root matrix is actually the given matrix* **C**.

Since the resolvent of the matrix **C** is

$$\mathbf{R} = (\lambda\mathbf{I} - \mathbf{C})^{-1} = \frac{1}{(\lambda-4)(\lambda-1)^2}\begin{bmatrix} \lambda^2 - 20\lambda + 145 & 6\,(\lambda-10) & 18 \\ 339 - 24\lambda & \lambda^2 + 4\lambda - 140 & 3\,(\lambda+14) \\ 33\lambda + 30 & 3\,(5\lambda+4) & (\lambda+2)^2 \end{bmatrix},$$

the given matrix is not diagonalizable. So we can not calculate Sylvester's auxiliary matrices and we have to use the resolvent method.

```
c = {{-14, 6, 0}, {-24, 10, 3}, {33, -15, 10}}
Rc = Simplify[Inverse[lambda*IdentityMatrix[3] - c]]
{{(145 - 20 lambda + lambda^2)/((-4 + lambda) (-1 + lambda)^2), (
  6 (-10 + lambda))/((-4 + lambda) (-1 + lambda)^2),
  18/((-4 + lambda) (-1 + lambda)^2)}, {(
  339 - 24 lambda)/((-4 + lambda) (-1 + lambda)^2), (-140 + 4 lambda +
    lambda^2)/((-4 + lambda) (-1 + lambda)^2), (
  3 (14 + lambda))/((-4 + lambda) (-1 + lambda)^2)}, {(
  30 + 33 lambda)/((-4 + lambda) (-1 + lambda)^2), -((
   3 (4 + 5 lambda))/((-4 + lambda) (-1 + lambda)^2)), (2 +
    lambda)^2/((-4 + lambda) (-1 + lambda)^2)}}
```

For future calculations, we need the expression for numerator:

```
RR[x_] = Simplify[
  Inverse[x*IdentityMatrix[3] - c]*(x - 1)^2 * (x - 4)]
```

First, we calculate the residue at eigenvalue $\lambda = 4$:

$$\mathbf{r}_4 = \operatorname*{Res}_{\lambda=4} \mathbf{R}\,\sqrt{\lambda} = \pm \begin{bmatrix} 9 & -4 & 2 \\ 27 & -12 & 6 \\ 18 & -8 & 4 \end{bmatrix},$$

```
R4 = FullSimplify[2*RR[4]/(4 - 1)^2]
```

Then we calculate the residue at $\lambda = 1$:

$$\mathbf{r}_1 = \operatorname*{Res}_{\lambda=1} \mathbf{R}\,\sqrt{\lambda} = \lim_{\lambda\to 1} \frac{\mathrm{d}}{\mathrm{d}\lambda}\,\mathbf{R}(\lambda)\,\sqrt{\lambda}\,(\lambda-1)^2$$
$$= \pm\frac{1}{2}\begin{bmatrix} -58 & 26 & -10 \\ -159 & 71 & -27 \\ -57 & 25 & -9 \end{bmatrix}.$$

```
R1 = FullSimplify[D[RR[x]*Sqrt[x]/(x - 4), x] /. x -> 1]
```

There are four matrix roots depending on sign of a square root, but we need only two of them because other two matrix roots will have opposite sign:

$$\mathbf{R}_1 = \mathbf{r}_1 + \mathbf{r}_4 = \frac{1}{2}\begin{bmatrix} -22 & 10 & -2 \\ -51 & 23 & -3 \\ 15 & -7 & 7 \end{bmatrix},$$
$$\mathbf{R}_2 = \mathbf{r}_1 - \mathbf{r}_4 = \frac{1}{2}\begin{bmatrix} -94 & 42 & -18 \\ -267 & 119 & -51 \\ -129 & 57 & -25 \end{bmatrix}.$$

We check with *Mathematica*:

```
r1 = R1 + R4
{{-11, 5, -1}, {-(51/2), 23/2, -(3/2)}, {15/2, -(7/2), 7/2}}
r2 = R1 - R4
{{-47, 21, -9}, {-(267/2), 119/2, -(51/2)}, {-(129/2), 57/2, -(25/2)}}
r1.r1 - c
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
r2.r2 - c
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Here is Matlab code to find square root of a nondiagonalizable matrix:

```
clear all; close all; clc;

% this script computes the square root of a nondiagonalizable matrix using
% the resolvent method
C = [-14 6 0;
     -24 10 3;
      33 -15 10];

syms t;

% compute the resolvent of C
R = inv(C-t*eye(3));

% we can display the result in a slightly more presentable form:
pretty(R)

% symbolically evaluate the residues of f(t)*R(t) at t=1 and t=4, where
% f(t) = sqrt(t)
r4 = simplify(R*sqrt(t)*(t-4));
r1 = diff(simplify(R*sqrt(t)*(t-1)^2), t);

```

```
% evaluate residues at the eigenvalues
res4 = subs(r4,4);
res1 = subs(r1,1);

SqC_1 = res1 + res4;
SqC_2 = res1 - res4;

% there are four roots of the matrix
pretty(SqC_1)
pretty(-SqC_1)
pretty(SqC_2)
pretty(-SqC_2)
```