AM 034 — Applied Mathematics - II

Brown University **Spring 2019**

Solutions to Homework, Set 2 **Due February 20**

2.1 (20 points) For given two 3×3 matrices $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$ and $\mathbf{S} = \begin{bmatrix} 4 & 2 & 5 \\ 5 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, calculate the matrix $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$. Then determine the eigenvalues and corresponding eivenvectors for matrices $\mathbf{A}$ and $\mathbf{B}$.

Solution. Since $\mathbf{A}$ is a diagonal matrix, its eigenvalues are on the main diagonal, so $\lambda_1 = 1$, $\lambda_2 = -3$, and $\lambda_3 = 1/2$. The corresponding eigenvectors are the unit vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then we find the inverse of the matrix $\mathbf{S}$:

$$\mathbf{S}^{-1} = \begin{bmatrix} 4 & 2 & 5 \\ 5 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 1 & 8 \\ 13 & -2 & -21 \\ -1 & 0 & 2 \end{bmatrix}.$$

Mathematica confirms:

```
S = {{4, 2, 5}, {5, 2, 1}, {2, 1, 3}}
Inverse[S]
Out[2]= {{-5, 1, 8}, {13, -2, -21}, {-1, 0, 2}}
```

Calculations show that

$$\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} -27 & -12 & -16 \\ 61 & 55/2 & 79/2 \\ -2 & -1 & -2 \end{bmatrix}.$$

The matrix $\mathbf{B}$ has the same eigenvalues as the diagonal matrix $\mathbf{A}$, but its eigenvectors are different. We check our calculations with *Mathematica:*

```
A = DiagonalMatrix[{1, -3, 1/2}]
B = Inverse[S].A.S
Out[4]= {{-27, -12, -16}, {61, 55/2, 79/2}, {-2, -1, -2}}
Eigenvalues[B]
Out[5]= {-3, 1, 1/2}
Eigenvectors[B]
Out[6]= {{-(1/2), 1, 0}, {5, -13, 1}, {4, -(21/2), 1}}
```

We can also check eigenvalues and eigenvectors with MATLAB:

```
>> B = [-27 -12 -16; 61 55/2 79/2; -2 -1 -2];
>> lambda = eig(B) % or to see eivenvectors as well:
>> [V,D] = eig(B)
```

So the matrix $\mathbf{B}$ has the same eigenvalues as the diagonal matrix $\mathbf{A}$: -3,1,1/2; however, their eigenvectors are different:

$$\mathbf{e}_{-3} = \langle -1, 2, 0 \rangle^T, \qquad \mathbf{e}_1 = \langle 5, -13, 1 \rangle^T, \qquad \mathbf{e}_{1/2} = \langle 8, 21, 2 \rangle^T.$$

Then we do a similar job with MuPad:

```
A := matrix(3, 3, [[1,0,0],[0,-3,0],[0,0,(1/2)]])
S := matrix(3,3, [[4,2,5],[5,2,1],[2,1,3]])
B := inverse(S)*A*S
linalg::eigenvalues(B)
Ev := linalg::eigenvectors(B)
Eigenvectors := 2*Ev[1][3][1], Ev[2][3][1], 2*Ev[3][3][1]
```

2.2 (20 points) For what vectors $\mathbf{b}$ does the linear system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a nontrivial solution?

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 3 & 2 \\ 2 & 1 & 4 \end{bmatrix}.$$

Solution. *Grading: For each part: +8 for attempting to solve homogeneous equation, +4 for correct* $\mathbf{y}$*, +6 for correct final answer (i.e. correct equation for components of* $\mathbf{b}$*).*

Solving the homogeneous equation $\mathbf{A}^T\mathbf{y} = \mathbf{0}$ for the transposed matrix $\mathbf{A}^T$, we get $\mathbf{y} = \langle -5, 8, 1 \rangle^T$. We check the answer with *Mathematica:*

```
NullSpace[Transpose[{{2, 5, 4}, {1, 3, 2}, {2, 1, 4}}]]
Out[2]= {{-5, 8, 1}}
```

MuPad codes:

```
A := matrix(3,3,[[2,5,4],[1,3,2],[2,1,4]])
det(A)
transpose(A)
```

Then the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution if and only if vector $\mathbf{b}$ is orthogonal to $\mathbf{y}$, that is, their dot product must be zero: $\mathbf{b}\cdot\mathbf{y} = 0$. Therefore, the components of $\mathbf{b} = \langle b_1, b_2, b_3\rangle^T$ should satisfy the equation:

$$-5\,b_1 + 8\,b_2 + b_3 = 0.$$

2.3 $(7 \times 4 + 12 = 40$ points) For each of the following matrices,

$$(\mathbf{a}) \quad \begin{bmatrix} -37 & -20 & -70 \\ 106 & 56 & 185 \\ -12 & -6 & -17 \end{bmatrix}, \qquad (\mathbf{b}) \quad \begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & -1 \\ 1 & 3 & -1 \end{bmatrix}, \qquad (\mathbf{c}) \quad \begin{bmatrix} 17 & 8 & 24 \\ -42 & -20 & -63 \\ 4 & 2 & 7 \end{bmatrix},$$

find $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ by performing the following steps:

(a) determine all eigenvalues and corresponding eigenvectors;

(b) Form a 3×3 matrix $\mathbf{S}$, where each column is an eigenvector;

(c) Calculate the diagonal matrix $e^{\mathbf{D}t}$ having on diagonal $e^{\lambda_j t}$, where λ_j $(j = 1, 2, 3)$ is an eigenvalue;

(d) Find $\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathbf{S}\, e^{\mathbf{D}t}\, \mathbf{S}^{-1}$.

(e) $(4 \times 3 = 12$ points) Show that $\mathbf{\Phi}(t)$ for each matrix is a solution of matrix differential equation subject to the initial condition:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Phi}(t) = \mathbf{A}\,\mathbf{\Phi}(t), \qquad \mathbf{\Phi}(0) = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. *Grading: For each part: +3 for using the right formulas for matrices of eigenvectors, +2 for finding correct inverse matrices, +1 for constructing the diagonal matrices $e^{\mathbf{D}t}$, +1 for correctly finding $e^{\mathbf{A}t}$*

(a) Since the eigenvalues of the matrix $\mathbf{A}$ are distinct real numbers: $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 3$, the corresponding eigenvectors are

$$\mathbf{v}_1 = \langle 5, -13, 1\rangle^T, \quad \mathbf{v}_{-2} = \langle 8, -21, 2\rangle^T, \quad \mathbf{v}_3 = \langle -1, 2, 0\rangle^T.$$

Then

$$\mathbf{S} = \begin{bmatrix} -1 & 8 & 5 \\ 2 & -21 & -13 \\ 0 & 2 & 1 \end{bmatrix} \qquad \Longrightarrow \qquad \mathbf{S}^{-1} = \begin{bmatrix} -5 & -2 & -1 \\ 2 & 1 & 3 \\ -4 & -2 & -5 \end{bmatrix}.$$

Now we find the exponential matrix

$$e^{\mathbf{A}t} = \mathbf{S} \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} \mathbf{S}^{-1}$$

$$= e^{-2t} \begin{bmatrix} 16 & 8 & 24 \\ -42 & -21 & -63 \\ 4 & 2 & 6 \end{bmatrix} + e^{3t} \begin{bmatrix} 5 & 2 & 1 \\ -10 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix} + e^{t} \begin{bmatrix} -20 & -10 & -25 \\ 52 & 26 & 65 \\ -4 & -2 & -5 \end{bmatrix}.$$

MuPad script:

```
A := matrix(3,3,[[-37,-20,-70],[106,56,185],[-12,-6,-17]])
linalg::eigenvalues(A)
Ev := linalg::eigenvectors(A)
map(Ev, op@op ,3)
S := matrix(3,3,[[-1,8,5],[2,-21,-13],[0,2,1]])
eDt := matrix(3,3,[[exp(3*t),0,0],[0,exp(-2*t),0],[0,0,exp(t)]])
eAt := S*eDt*inverse(S)
```

Then we differentiate

$$\frac{\mathrm{d}}{\mathrm{d}t}\, e^{\mathbf{A}t} = -2\, e^{-2t} \begin{bmatrix} 16 & 8 & 24 \\ -42 & -21 & -63 \\ 4 & 2 & 6 \end{bmatrix} + 3\, e^{3t} \begin{bmatrix} 5 & 2 & 1 \\ -10 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix} + e^{t} \begin{bmatrix} -20 & -10 & -25 \\ 52 & 26 & 65 \\ -4 & -2 & -5 \end{bmatrix}.$$

We compare right hand side with the products:

$$\mathbf{A} \begin{bmatrix} 16 & 8 & 24 \\ -42 & -21 & -63 \\ 4 & 2 & 6 \end{bmatrix} = (-2) \begin{bmatrix} 16 & 8 & 24 \\ -42 & -21 & -63 \\ 4 & 2 & 6 \end{bmatrix};$$

Mathematica confirms:

```
A.{{16, 8, 24}, {-42, -21, -63}, {4, 2, 6}} +
 2*{{16, 8, 24}, {-42, -21, -63}, {4, 2, 6}}
Out[10]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

$$\mathbf{A} \begin{bmatrix} -20 & -10 & -25 \\ 52 & 26 & 65 \\ -4 & -2 & -5 \end{bmatrix} = \begin{bmatrix} -20 & -10 & -25 \\ 52 & 26 & 65 \\ -4 & -2 & -5 \end{bmatrix}$$

Mathematica confirms:

```
A.{{-20, -10, -25}, {52, 26, 65}, {-4, -2, -5}}
Out[11]= {{-20, -10, -25}, {52, 26, 65}, {-4, -2, -5}}
```

$$\mathbf{A} \begin{bmatrix} 5 & 2 & 1 \\ -10 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix} = 3 \begin{bmatrix} 5 & 2 & 1 \\ -10 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Mathematica confirms:

```
A.{{5, 2, 1}, {-10, -4, -2}, {0, 0, 0}} -
 3*{{5, 2, 1}, {-10, -4, -2}, {0, 0, 0}}
Out[12]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

(b) Since two eigenvalues of the given matrix are complex conjugate, their eigenvectors are also complex conjugate. Therefore, it is enough to consider only one complex eigenvalue:

$$\lambda_5 = 5, \quad \mathbf{v}_5 = \langle 21, 5, 6\rangle^T, \qquad \lambda_c = -1 + \mathbf{j}, \quad \mathbf{v}_c = \langle -3 - 2\mathbf{j}, 1 + \mathbf{j}, 1\rangle^T.$$

Mathematica confirms:

```
A = {{2, 3, 8}, {1, 2, -1}, {1, 3, -1}}
Eigenvalues[A]
Out[18]= {5, -1 + I, -1 - I}
Eigenvectors[A]
Out[19]= {{21, 5, 6}, {-3 - 2 I, 1 + I, 1}, {-3 + 2 I, 1 - I, 1}}
S = {{21, -3 - 2*I, -3 + 2*I}, {5, 1 + I, 1 - I}, {6, 1, 1}}
Inverse[S]
Out[21]= {{1/37, 2/37, 1/37}, {-(3/37) - I/74, -(6/37) - (39 I)/74,
  31/74 + (18 I)/37}, {-(3/37) + I/74, -(6/37) + (39 I)/74,
  31/74 - (18 I)/37}}
```

Then the matrix of eigenvectors becomes

$$\mathbf{S} = \begin{bmatrix} 21 & -3-2\mathbf{j} & -3+2\mathbf{j} \\ 5 & 1+\mathbf{j} & 1-\mathbf{j} \\ 6 & 1 & 1 \end{bmatrix} \qquad \Longrightarrow \qquad \mathbf{S}^{-1} = \frac{1}{74}\begin{bmatrix} 2 & 4 & 2 \\ -6-\mathbf{j} & -12-39\mathbf{j} & 31+36\mathbf{j} \\ -6+\mathbf{j} & -12+39\mathbf{j} & 31-36\mathbf{j} \end{bmatrix}$$

Now we find the exponential matrix

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{S} \begin{bmatrix} e^{5t} & 0 & 0 \\ 0 & e^{-t+\mathbf{j}t} & 0 \\ 0 & 0 & e^{-t-\mathbf{j}t} \end{bmatrix} \mathbf{S}^{-1} \\ &= \frac{1}{37}\, e^{5t} \begin{bmatrix} 21 & 42 & 21 \\ 5 & 10 & 5 \\ 6 & 12 & 6 \end{bmatrix} + \frac{\cos t}{37}\, e^{-t} \begin{bmatrix} 16 & -42 & -21 \\ -5 & 27 & -5 \\ -6 & -12 & 31 \end{bmatrix} + \frac{\sin t}{37}\, e^{-t} \begin{bmatrix} -15 & -141 & 170 \\ 7 & 51 & -67 \\ 1 & 39 & -36 \end{bmatrix}. \end{aligned}$$

We check with *Mathematica:*

```
A = {{2, 3, 8}, {1, 2, -1}, {1, 3, -1}}
MatrixExp[A*t]
S = {{21, -3 - 2*I, -3 + 2*I}, {5, 1 + I, 1 - I}, {6, 1, 1}}
SI = Inverse[S]
dd = {{Exp[5*t], 0, 0}, {0, Exp[-t]*(Cos[t] + I*Sin[t]), 0}, {0, 0,
    Exp[t]*(Cos[t] - I*Sin[t])}}
ComplexExpand[S.dd.SI]
```

Then we differentiate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\, e^{\mathbf{A}t} &= \frac{5}{37}\, e^{5t} \begin{bmatrix} 21 & 42 & 21 \\ 5 & 10 & 5 \\ 6 & 12 & 6 \end{bmatrix} \\ &\quad - \frac{\sin t + \cos t}{37}\, e^{-t} \begin{bmatrix} 16 & -42 & -21 \\ -5 & 27 & -5 \\ -6 & -12 & 31 \end{bmatrix} + \frac{\cos t - \sin t}{37}\, e^{-t} \begin{bmatrix} -15 & -141 & 170 \\ 7 & 51 & -67 \\ 1 & 39 & -36 \end{bmatrix} \\ &= \frac{5}{37}\, e^{5t} \begin{bmatrix} 21 & 42 & 21 \\ 5 & 10 & 5 \\ 6 & 12 & 6 \end{bmatrix} + \frac{\cos t}{37}\, e^{-t} \begin{bmatrix} -31 & -99 & 191 \\ 12 & 24 & -62 \\ 7 & 51 & -5 \end{bmatrix} + \frac{\sin t}{37}\, e^{-t} \begin{bmatrix} -1 & 183 & 149 \\ -2 & -78 & 72 \\ 5 & -27 & 5 \end{bmatrix}.\end{aligned}$$

We compare right hand side with the products:

$$\mathbf{A} \begin{bmatrix} 21 & 42 & 21 \\ 5 & 10 & 5 \\ 6 & 12 & 6 \end{bmatrix} = 5 \begin{bmatrix} 21 & 42 & 21 \\ 5 & 10 & 5 \\ 6 & 12 & 6 \end{bmatrix};$$

```
A.{{21, 42, 21}, {5, 10, 5}, {6, 12, 6}}/5
Out[25]= {{21, 42, 21}, {5, 10, 5}, {6, 12, 6}}
```

$$\mathbf{A} \begin{bmatrix} 16 & -42 & -21 \\ -5 & 27 & -5 \\ -6 & -12 & 31 \end{bmatrix} = \begin{bmatrix} -31 & -99 & 191 \\ 12 & 24 & -62 \\ 7 & 51 & -5 \end{bmatrix};$$

```
A.{{16, -42, -21}, {-5, 27, -5}, {-6, -12, 31}}
Out[26]= {{-31, -99, 191}, {12, 24, -62}, {7, 51, -67}}
```

$$\mathbf{A} \begin{bmatrix} -15 & -141 & 170 \\ 7 & 51 & -67 \\ 1 & 39 & -36 \end{bmatrix} = \begin{bmatrix} -1 & 183 & 149 \\ -2 & -78 & 72 \\ 5 & -27 & 5 \end{bmatrix}.$$

```
A.{{-15, -141, 170}, {7, 51, -67}, {1, 39, -36}}
Out[26]= {{-1, 183, -149}, {-2, -78, 72}, {5, -27, 5}}
```

(c) While the characteristic polynomial $\det\left(\lambda\mathbf{I} - \mathbf{A}\right) = \left(\lambda - 1\right)^2 \left(\lambda - 2\right)$ has a double eigenvalue $\lambda = 1$, the matrix is not defective because there are two eigenvectors corresponding to this eigenvalue:

$$\lambda_1 = 1, \quad \mathbf{v}_{11} = \langle -3, 0, 2\rangle^T, \quad \lambda_2 = 1, \quad \mathbf{v}_{12} = \langle -1, 2, 0\rangle^T, \quad \lambda_3 = 2, \quad \mathbf{v}_2 = \langle 8, -21, 2\rangle^T.$$

Then the matrix of eigenvectors becomes

$$\mathbf{S} = \begin{bmatrix} 8 & -3 & -1 \\ -21 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \qquad \Longrightarrow \qquad \mathbf{S}^{-1} = \begin{bmatrix} 2 & 1 & 3 \\ -2 & -1 & 5/2 \\ 21 & 11 & 63/2 \end{bmatrix}.$$

```
A = {{17, 8, 24}, {-42, -20, -63}, {4, 2, 7}}
ei = Eigenvectors[B]
S = Transpose[{ei[[1]], ei[[2]], ei[[3]]}]
Inverse[S]
Out= {{2, 1, 3}, {-2, -1, -(5/2)}, {21, 11, 63/2}}
```

Now we find the exponential matrix

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{S} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} \mathbf{S}^{-1} \\ &= e^t \begin{bmatrix} -15 & -8 & -24 \\ 42 & 22 & 63 \\ -4 & -2 & -5 \end{bmatrix} + e^{2t} \begin{bmatrix} 16 & 8 & 24 \\ -42 & -21 & -63 \\ 4 & 2 & 6 \end{bmatrix}. \end{aligned}$$

Then we differentiate

$$\frac{\mathrm{d}}{\mathrm{d}t}\, e^{\mathbf{A}t} = e^t \begin{bmatrix} -15 & -8 & -24 \\ 42 & 22 & 63 \\ -4 & -2 & -5 \end{bmatrix} + 2\, e^{2t} \begin{bmatrix} 16 & 8 & 24 \\ -42 & -21 & -63 \\ 4 & 2 & 6 \end{bmatrix}.$$

We compare right hand side with the products:

$$\mathbf{A} \begin{bmatrix} -15 & -8 & -24 \\ 42 & 22 & 63 \\ -4 & -2 & -5 \end{bmatrix} = \begin{bmatrix} -15 & -8 & -24 \\ 42 & 22 & 63 \\ -4 & -2 & -5 \end{bmatrix}$$

```
A = {{17, 8, 24}, {-42, -20, -63}, {4, 2, 7}}
et = {{-15, -8, -24}, {42, 22, 63}, {-4, -2, -5}}
A.et
Out= {{-15, -8, -24}, {42, 22, 63}, {-4, -2, -5}}
```

$$\mathbf{A} \begin{bmatrix} 16 & 8 & 24 \\ -42 & -21 & -63 \\ 4 & 2 & 6 \end{bmatrix} = 2 \begin{bmatrix} 16 & 8 & 24 \\ -42 & -21 & -63 \\ 4 & 2 & 6 \end{bmatrix}.$$

```
A = {{17, 8, 24}, {-42, -20, -63}, {4, 2, 7}}
e2t = {{16, 8, 24}, {-42, -21, -63}, {4, 2, 6}}
A.e2t
Out[31]= {{32, 16, 48}, {-84, -42, -126}, {8, 4, 12}}
B.e2t - 2*e2t
Out[32]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

2.4 (10 + 10 = 20 points) Given the matrices **A** and **B** as shown below. Find eigenvalues of their products **A B** and **B A** and determine their corresponding eigenvectors. Is matrix **B A** diagonalizable?

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 3 & 2 \\ 2 & 5 & 1 & -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ -3 & 1 \\ 2 & 3 \end{bmatrix}.$$

Solution. *Grading: +10 for finding eigenvalues and eigenvectors for the product* **A B**, *and correspondingly +10 for doing the same job for* **B A**

First we calculate the products:

$$\mathbf{T}_1 = \mathbf{A}\,\mathbf{B} = \begin{bmatrix} -2 & 7 \\ -6 & 15 \end{bmatrix} \qquad \text{and} \qquad \mathbf{T}_2 = \mathbf{B}\,\mathbf{A} = \begin{bmatrix} 4 & 3 & 7 & 3 \\ 5 & 16 & 0 & -5 \\ -1 & 8 & -8 & -7 \\ 8 & 13 & 9 & 1 \end{bmatrix}.$$

The eigenvalues and corresponding eigenvectors for these matrices are

$$\lambda_1 = 1, \quad \mathbf{v}_1 = \langle 7, 3\rangle^T, \qquad \lambda_{12} = 12, \quad \mathbf{v}_{12} = \langle 1, 2\rangle^T,$$

and for the matrix $\mathbf{T}_2$

$$\lambda_{19} = 1, \quad \mathbf{v}_1 = \langle 17, 2, -18, 23\rangle^T, \qquad \lambda_{12} = 12, \quad \mathbf{v}_{12} = \langle 4, 5, -1, 8\rangle^T,$$

$$\lambda_0 = 0, \quad \mathbf{v}_0 = \langle -9, 5, 0, 7\rangle^T, \quad \mathbf{v}_{00} = \langle -16, 5, 7, 0\rangle^T.$$

Therefore the geometrical multiplicity of $\lambda_0 = 0$ is 2, the same as its algebraic one. All other eigenvalues are simple, so there is no difference in their multiplicities. Both matrices, **A B** and **B A**, are diagonalizable.

MuPad script:

```
A := matrix(2,4,[[1,-1,3,2],[2,5,1,-1]])
B := matrix(4,2,[[2,1],[-1,3],[-3,1],[2,3]])
T1 := A*B
T2 := B*A
linalg::eigenvectors(T1)
linalg::eigenvectors(T2)
```

Mathematca codes:

```
T2 = {{4, 3, 7, 3}, {5, 16, 0, -5}, {-1, 8, -8, -7}, {8, 13, 9, 1}}
NullSpace[T2]
Out[2]= {{-9, 5, 0, 7}, {-16, 5, 7, 0}}
```