AM 034 — Applied Mathematics - II

Brown University **Spring 2019**

Solutions to Homework, Set 1 **Due February 13**

1.1 (30 points) Find a first order system that is equivalent to the second order equation named after the German scientist Georg Duffing (1861–1944)

$$\ddot{y} + 2\eta\,\dot{y} + y + \varepsilon\,y^3 = \cos(\omega t),$$

where the dots denote differentiation of $y(t)$ with respect t: $\dot{y} = \mathrm{d}y/\mathrm{d}t$. Here $\eta \geq 0$ and ε are physical parameters, and ω is given frequency of excitation. In general, the Duffing equation does not admit an exact symbolic solution.

Solution. Upon introduction of two new dependent variables $x_1 = y$ and $x_2 = \dot{x}_1 = \dot{y}$, we reduce the given system of differential equations to the following one:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - \varepsilon\,x_1^3 - 2\eta\,x_2 + \cos(\omega t). \end{aligned}$$

1.2 (40 points) The initial value problem for the vector differential equation

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{f}(t,\mathbf{u}), \qquad \mathbf{u}(0) = \mathbf{u}_0,$$

where $\mathbf{u}$ is unknown column-vector and $\mathbf{f}$ is a given column vector, is equivalent to the integral equation

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{f}(s,\mathbf{u}(s))\,\mathrm{d}s.$$

Its solution can be found as the limit $\mathbf{u}(t) = \lim_{n\to\infty} \boldsymbol{\phi}_n(t)$, where the sequence of vector functions is defined recursively by

$$\boldsymbol{\phi}_0 = \mathbf{u}_0, \qquad \boldsymbol{\phi}_{n+1}(t) = \mathbf{u}_0 + \int_0^t \mathbf{f}\left(s,\boldsymbol{\phi}_n(s)\right)\mathrm{d}s, \qquad n = 0,1,2,\ldots.$$

Consider the initial value problem for the homogeneous Duffing equation:

$$\ddot{y} + 2\eta\,\dot{y} + y + \varepsilon\,y^3 = 0, \qquad y(0) = d, \quad \dot{y}(0) = v,$$

where $d = 1$ is the initial displacement and $v = -1$ is the initial velocity.

Convert the above initial value problem to the system of first order equations for the following numerical values: $\eta = 3$ and $\varepsilon = 1$. Then using Picard's iteration, find the first four iterative terms $\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), \boldsymbol{\phi}_3(t), \boldsymbol{\phi}_4(t)$ for the Duffing system of equations. Finally, determine the corresponding four approximations to the solution $y(t)$ of the original second order Duffing equation.

Solution. We rewrite the Picard iterations explicitly:

$$\boldsymbol{\phi}_0 = \begin{bmatrix} d \\ v \end{bmatrix}, \qquad \boldsymbol{\phi}_{n+1}(t) = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \begin{bmatrix} d \\ v \end{bmatrix} + \int_0^t \begin{bmatrix} y(s) \\ -x(s) - \varepsilon\, x^3(x) - 2\eta\, y(s) \end{bmatrix}_n \mathrm{d}s, \qquad n = 0, 1, 2, \ldots.$$

We show each iteration separately.

$$\begin{aligned}
\boldsymbol{\phi}_1 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \int_0^t \begin{bmatrix} -1 \\ 6-1-1 \end{bmatrix} \mathrm{d}s = \begin{bmatrix} 1-t \\ 4t-1 \end{bmatrix}, \\
\boldsymbol{\phi}_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \int_0^t \begin{bmatrix} 4s-1 \\ 6(1-4s)-(1-s)-(1-s)^3 \end{bmatrix} \mathrm{d}s \\
&= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \int_0^t \begin{bmatrix} 4s-1 \\ [4-20s-3s^2+s^3] \end{bmatrix} \mathrm{d}s = \begin{bmatrix} 1-t+2t^2 \\ -1+4t-10t^2-t^3+\frac{t^4}{4} \end{bmatrix}, \\
\boldsymbol{\phi}_3 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \int_0^t \begin{bmatrix} -1+4s-10s^2-s^3+\frac{s^4}{4} \\ -(1-s+2s^2)-(1-s+2s^2)^3-6\left(-1+4s-10s^2-s^3+\frac{s^4}{4}\right) \end{bmatrix} \mathrm{d}s \\
&= \begin{bmatrix} 1-t+2t^2-\frac{10}{3}\,t^3-\frac{t^4}{4}+\frac{t^5}{20} \\ -1+4t-10t^2+\frac{49}{3}\,t^3+\frac{19}{4}\,t^4-\frac{39}{10}\,t^5+2\,t^6-\frac{8}{7}\,t^7 \end{bmatrix}, \\
\boldsymbol{\phi}_4 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \int_0^t \begin{bmatrix} -1+4s-10s^2+\frac{49}{3}\,s^3+\frac{19}{4}\,s^4-\frac{39}{10}\,s^5+2\,s^6-\frac{8}{7}\,s^7 \\ 4-20t+\cdots \end{bmatrix} \mathrm{d}s \\
&= \begin{bmatrix} 1-t+2t^2-\frac{10}{3}\,t^3+\frac{49}{12}\,t^4+\frac{19}{20}\,t^5-\frac{13}{20}\,t^6+\frac{2}{7}\,t^7-\frac{1}{7}\,t^8 \\ -1+4t-10t^2+\frac{49}{3}\,t^3-\frac{215}{12}\,t^4-\frac{131}{10}\,t^5+\frac{279}{20}\,t^6-\frac{5357}{420}\,t^7+\cdots \end{bmatrix}.
\end{aligned}$$

1.3 (30 points) Consider a third order constant coefficient differential equation

$$y''' - 3\,y'' + 7\,y' - 2\,y = e^{-|t|}\,\cos 2t.$$

Convert this equation into a system of first order equations and rewrite it in matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{f}(t),$$

clearly identifying the 3-by-3 constant matrix $\mathbf{A}$ and 3-column vector $\mathbf{f}$.

Solution. Upon introducing three new dependent variables $x_1 = y$, $x_2 = \dot{y} = \dot{x}_1$, and $x_3 = \ddot{y} = \dot{x}_2$, we obtain

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= 3\,x_3 - 7\,x_2 + 2\,x_1 + e^{-|t|}\,\cos 2t
\end{aligned}$$

which we rewrite in matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -7 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^{-|t|}\cos 2t \end{bmatrix}.$$