AM 034 — Applied Mathematics - II

Brown University **Fall 2021**

Solutions to Homework, Set 2 **Due September 29**

2.1 (30 points) Consider the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \end{bmatrix}.$$

(a) (10 points) Write down an integral equation that is equivalent to the given initial value problem.

(b) (20 points) Using Picard's approximation, find four-term approximation to the given initial value problem.

Solution.

(a)

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \end{bmatrix} + \int_0^t \begin{bmatrix} -1 * x(s) & 2 * y(s) \\ 3 * x(s) & 4 * y(s) \end{bmatrix} \mathbf{d}s.$$

(b)

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}_{m+1} = \begin{bmatrix} 7 \\ -7 \end{bmatrix} + \int_0^t \begin{bmatrix} -1 * x(s) + 2 * y(s) \\ 3 * x(s) + 4 * y(s) \end{bmatrix}_m \mathbf{d}s$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}_0 = \begin{bmatrix} 7 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}_1 = \begin{bmatrix} 7 \\ -7 \end{bmatrix} + \int_0^t \begin{bmatrix} -1 * 7 + 2 * -7 \\ 3 * 7 + 4 * -7 \end{bmatrix} \mathbf{d}x = \begin{bmatrix} 7 - 21t \\ -7 - 7t \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}_2 = \begin{bmatrix} 7 \\ -7 \end{bmatrix} + \int_0^t \begin{bmatrix} -1 * (7 - 21t) + 2 * (-7 - 7t) \\ 3 * (7 - 21t) + 4 * (-7 - 7t) \end{bmatrix} \mathbf{d}x = \begin{bmatrix} 7 - 21t + \frac{7}{2}t^2 \\ -7 - 7t - \frac{91}{2}t^2 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}_3 &= \begin{bmatrix} 7 \\ -7 \end{bmatrix} + \int_0^t \begin{bmatrix} -1 * (7 - 21t + \frac{7}{2}t^2) + 2 * (-7 - 7t - \frac{91}{2}t^2) \\ 3 * (7 - 21t + \frac{7}{2}t^2) + 4 * (-7 - 7t - \frac{91}{2}t^2) \end{bmatrix} \mathbf{d}t \\ &= \begin{bmatrix} 7 - 21t + \frac{7}{2}t^2 - \frac{189}{6}t^3 \\ -7 - 7t - \frac{91}{2}t^2 - \frac{343}{6}t^3 \end{bmatrix} \end{aligned}$$

This has four terms, so we are done.

2.2 (50 points) Consider a simple pendulum equation

$$\ddot{\theta} + \omega^2 \sin\theta = 0, \qquad \theta(0) = \frac{\pi}{3}, \quad \dot{\theta}(0) = -\frac{1}{10}.$$

(a) (10 points) Convert the given initial value problem to a system of first order differential equations suitable for Picard's iteration (so the right-hand side is a polynomial).

(b) (10 points) Write down an integral equation that is equivalent to the given initial value problem.

(c) (10 points) Set up a Picard iteration scheme based on part (b).

(d) (10 points) Perform three Picard's iterations to find a polynomial approximation to $\theta(t)$.

(e) (10 points) On interval [0, 6.0], plot your Picard's approximation along with the solution curve obtained upon application of a solution solver to the given initial value problem.

Solution.

(a)

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \\ \sin(\theta(t)) \\ \cos(\theta(t)) \end{bmatrix},$$

$$\frac{\mathbf{d}}{\mathbf{d}t} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} y_2(t) \\ -\omega^2 y_3(t) \\ y_2(t) * y_4(t) \\ -y_2(t) * y_3(t) \end{bmatrix}, \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{bmatrix} = \begin{bmatrix} \frac{\pi}{3} \\ \frac{-1}{10} \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}.$$

(b)

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} \frac{\pi}{3} \\ \frac{-1}{10} \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} + \int_0^t \begin{bmatrix} y_2(t) \\ -\omega^2 y_3(t) \\ y_2(t) * y_4(t) \\ -y_2(t) * y_3(t) \end{bmatrix} \mathbf{d}t$$

(c)

$$\begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \\ \phi_4(t) \end{bmatrix}_{m+1} = \begin{bmatrix} \frac{\pi}{3} \\ \frac{-1}{10} \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} + \int_0^t \begin{bmatrix} \phi_2(t) \\ -\omega^2 \phi_3(t) \\ \phi_2(t)\phi_4(t) \\ -\phi_2(t)\phi_3(t) \end{bmatrix}_m \mathbf{d}t, \qquad \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}_0 = \begin{bmatrix} \frac{\pi}{3} \\ \frac{-1}{10} \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

(d) we perform a few iterations starting with the initial conditions:

$$\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{bmatrix}_0 = \begin{bmatrix}\frac{pi}{3}\\ \frac{-1}{10}\\ \frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{bmatrix}$$

Then the next one becomes

$$\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{bmatrix}_1 = \begin{bmatrix}\frac{pi}{3}\\ \frac{-1}{10}\\ \frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{bmatrix} + \int_0^t \begin{bmatrix}\frac{-1}{10}\\ -\omega^2\frac{\sqrt{3}}{2}t\\ \frac{-1}{10}*\frac{1}{2}\\ \frac{1}{2}*\frac{\sqrt{3}}{2}\end{bmatrix}\,\mathbf{d}t = \begin{bmatrix}\frac{\pi}{3}-\frac{t}{10}\\ \frac{-1}{10}-\frac{\sqrt{3}*\omega^2*t}{2}\\ \frac{\sqrt{3}}{2}-\frac{t}{20}\\ \frac{1}{2}+\frac{\sqrt{3}t}{20}\end{bmatrix}$$

The second iteration is

$$\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{bmatrix}_2 = \begin{bmatrix}\frac{\pi}{3}-\frac{t}{10}-\frac{\sqrt{3}*\omega^2*t^2}{4}\\ \frac{-1}{10}-\frac{\sqrt{3}*\omega^2*t}{2}+\frac{\omega^2t^2}{40}\\ \frac{\sqrt{3}}{2}-\frac{t}{20}-\frac{t^2\sqrt{3}}{400}-\frac{\sqrt{3}\omega^2t^2}{8}-\frac{\omega^2t^3}{40}\\ \frac{1}{2}+\frac{\sqrt{3}t}{20}-\frac{t^2}{400}-\frac{3\omega^2t^2}{8}-\frac{\sqrt{3}\omega^2t^3}{40}\end{bmatrix}$$

Extracting the first component that approximate the requires function $\theta(t)$, we get

$$[\phi_1]_3 = \frac{\pi}{3} - \frac{t}{10} - \frac{\sqrt{3}*\omega^2*t^2}{4} + \frac{t^3\omega^2}{120}$$

We verify our answer with *Mathematica*:

```
AsymptoticDSolveValue[{theta''[t] + omega^2*Sin[theta[t]] == 0,
  theta[0] == Pi/3, theta'[0] == -1/10}, theta[t], {t, 0, 6}]
\[Pi]/3 - t/10 - 1/4 Sqrt[3] omega^2 t^2 + (
 omega^2 t^3)/120 + ((Sqrt[3] omega^2 + 50 Sqrt[3] omega^4) t^4)/4800
```

$\begin{bmatrix}\phi_2\\ \phi_3\\ \phi_4\end{bmatrix}_3$ **can be calculated and seen by running the** MATLAB **code below. This code was also used to calculate** $[\phi_1]_3$

```
syms t s w

expand(pi/3 + int(-1/10-sqrt(3)*w^2*s/2+ w^2*s^2/40,s,[0,t]))
expand(-1/10+ int(-w^2*(sqrt(3)/2 - s/20 -s^2*sqrt(3)/400-...
sqrt(3)*w^2*s^2/8-w^2*s^3/40),s,[0,t]))
expand(sqrt(3)/2+ int(-(-1/10-sqrt(3)*w^2*s/2+ w^2*s^2/40)...
*(1/2 +sqrt(3)/20*s-s^2/400+3*w^2*s^2/8-sqrt(3)*w^2*s^2/120),s,[0,t]))
expand(1/2+ int((-1/10-sqrt(3)*w^2*s/2+ w^2*s^2/40)...
*(sqrt(3)/2 - s/20 -s^2*sqrt(3)/400-sqrt(3)*w^2*s^2/8-w^2*s^3/40),s,[0,t]))
```

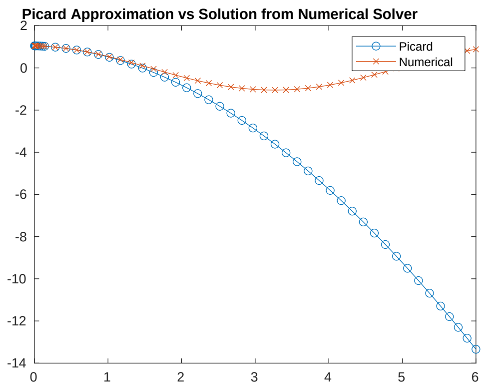

Figure 1: Plot with MATLAB.

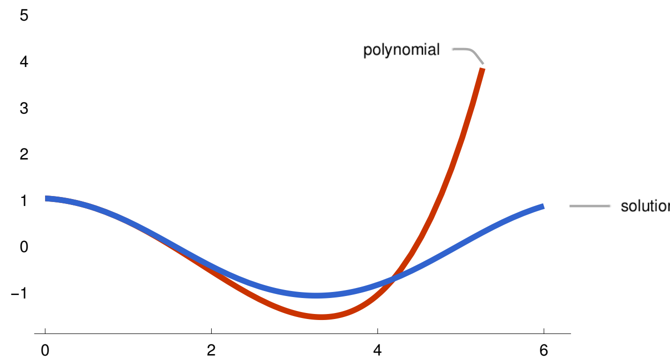

Figure 2: Plot with *Mathematica*.

(e)
```
%numerically solve pendulum equation. Let omega = 1
num_approx = @(t,y) [y(2); -sin(y(1))];
init_cond_b = [pi/3, -1/10];
t_interval = [0,6.0];
[t,y_num_approx] = ode45(num_approx,t_interval , init_cond_b);

%define Picard's polynomial approximation.
picard_approx = @(t) pi/3 - t/10 -sqrt(3)*t.^2/4 + t.^3/120;

%call function to get values for Picard approximation
y_picard_approx = picard_approx(t);

%plot numerical approximation and Picard approximation together
plot(t,y_picard_approx,'-o','DisplayName','Picard')
hold on
plot(t,y_num_approx(:,1),'-x','DisplayName','Numerical')
title("Picard Approximation vs Solution from Numerical Solver")
legend
```

Now we plot with *Mathematica:*

```
poly[t_] = (\[Pi]/3 - t/10 - 1/4 Sqrt[3] omega^2 t^2 + (omega^2 t^3)/
    120 + ((Sqrt[3] omega^2 + 50 Sqrt[3] omega^4) t^4)/4800) /.
  omega -> 1
s = NDSolve[{theta''[t] + Sin[theta[t]] == 0, theta[0] == Pi/3,
   theta'[0] == -1/10}, theta, {t, 0, 6}]
Plot[{Callout[poly[t], "polynomial", Above],
  Callout[Evaluate[{theta[t]} /. s], "solution"]}, {t, 0, 6},
 PlotTheme -> "Web"]
```

2.3 (20 points) Suppose you are given a matrix with variable entries:

$$\mathbf{A}(t) = \begin{bmatrix} t & t^2 \\ 1/t & 2-t \end{bmatrix}.$$

Find the derivative of its square $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{A}^2(t)$. Does a regular power rule hold ?

Solution. We have two options: either multiply matrix $\mathbf{A}$ by itself or use the power rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{A}^2(t) = \frac{\mathrm{d}\mathbf{A}(t)}{\mathrm{d}t}\,\mathbf{A}(t) + \mathbf{A}(t)\,\frac{\mathrm{d}\mathbf{A}(t)}{\mathrm{d}t}.$$

Since the derivative of $\mathbf{A}$ is easy to obtain

$$\frac{\mathrm{d}\mathbf{A}(t)}{\mathrm{d}t} = \begin{bmatrix} 1 & 2t \\ -1/t^2 & -1 \end{bmatrix}$$

```
D[{{t, t^2}, {1/t, 2 - t}}, t]
{{1, 2 t}, {-(1/t^2), -1}}
```

Then we just multiply two matrices and add the result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{A}^2(t) = \begin{bmatrix} 1-2t & 4t \\ -2/t^2 & 2t-3 \end{bmatrix}$$

Mathematica code:

```
A[t_] = {{t, t^2}, {1/t, 2 - t}}
D[Simplify[A[t].A[t]], t]
{{1 + 2 t, 4 t}, {-(2/t^2), -3 + 2 t}}
```