AM 034 — Applied Mathematics - II

Brown University **Fall 2021**

Solutions to Homework, Set 1 **Due September 22**

1.1 (30 points) Suppose that a projectile is launched into the air from ground with initial velocity v_0 and angle α with the horizontal plane. Assuming that there is no air resistance, the model for its movement is as follows

$$\begin{aligned} \ddot{x} &= 0, \qquad & x(0) = 0, \quad \dot{x}(0) = v_0 \cos\alpha, \\ \ddot{y} &= -g; \qquad & y(0) = 0, \quad \dot{y}(0) = v_0 \sin\alpha. \end{aligned}$$

Here $g \approx 9.81$ is the acceleration due to gravity and overdot denotes the derivative with respect to time t.

(a) (5 points) Show that the projectile's trajectory is a parabola.

(b) (5 points) Confirm part (a) by plotting several trajectories on an $x - y$ plane.

(c) (10 points) Let (x_m, y_m) be coordinates of the vertex corresponding to the parabola solution curve, so y_m is the maximum height attained and x_m is the corresponding horizontal coordinate. Plot the curve with coordinates (x_m, y_m) for different launch angles α.

(d) (10 points) Find the formula for the range R (the longest distance projectile travels) and determine the launch angle of the largest range.

Solution.

(a) (10 points) The trajectory is defined by parabola:

$$y = x\,\tan\alpha - x^2 \frac{g}{2\,v_0^2 \cos^2\alpha}$$

because

$$x = v_0 t\,\cos\alpha, \qquad y = v_0 t\,\sin\alpha - \frac{gt^2}{2}.$$

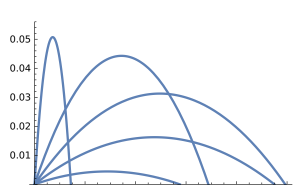

Figure 1: Projectile's trajectories plotted with Mathematica.

(c) (10 points)

$$x_m = \frac{v_0^2}{2g}\,\sin(2\alpha)\,, \qquad y_m = \frac{v_0^2}{2g}\,\sin^2\alpha$$

Mathematica code:

```
ParametricPlot[{Sin[2*a], (Sin[a])^2}, {a, 0, Pi/2},
 PlotTheme -> "Business"]
```

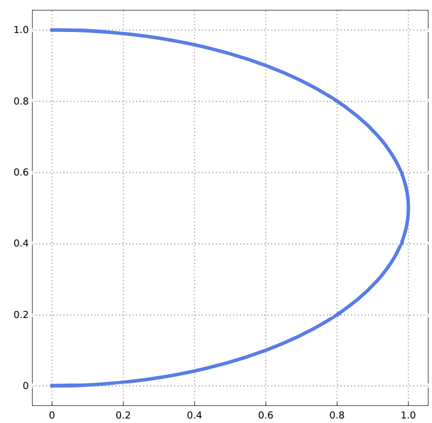

Figure 2: Vertex plot using Mathematica.

(d) (10 points) The range R (the longest distance projectile travels) is

$$R = \frac{v_0^2}{g}\,\sin(2\alpha)\,.$$

1.2 (40 points) Consider the second order constant coefficient differential equation

$$2\,x'' + 5\,x' - 3\,x(t) = 0, \qquad x(0) = 3, \quad x'(0) = -2.$$

(a) (10 points)
Convert this equation into a system of two equations of first order and rewrite it in matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x},$$

(b) (10 points) Upon introducing an auxiliary dependent variable $y_3(t) = x''(t)$, convert this equation into a system of three equations of first order and rewrite it in matrix form:

$$\dot{\mathbf{y}} = \mathbf{B}\,\mathbf{y},$$

clearly identifying the 3-by-3 constant matrix $\mathbf{B}$ and 3-column vector $\mathbf{y}$.

(c) (10 points) Using a numerical solver, plot on the interval [0,3] solution curves from parts (a) and (b). Do you detect any discrepancy ?

(d) (10 points) On interval [0,3], plot the difference of functions $2\,y_3(t) + 5\,x'(t) - 3\,x(t)$ in order to control computation of the differential equation.

Solution. Upon introducing new dependent variables

$$x_1(t) = x(t), \qquad x_2(t) = x'(t)$$

we transfer the given initial value problem into a vector problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3/2 & -5/2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Now we plot solutions in two previous parts with MATLAB:

```
t_interval = [0,3];
%
%initial conditions
init_cond_a = [3, -2]';
%system of equation
f_2a = @(t,y) [y(2); 3/2*y(1)-5/2*y(2)];
%calculate solution
[t,y_2a] = ode45(f_2a,t_interval , init_cond_a);
%
subplot(1,2,1)
plot(t, y_2a(:,1),'b','linewidth',3);
title('Solution from 1.2a')
%repeat
f_2b = @(t,y) [y(2); y(3); -15/4*y(1) + 31/4*y(2)];
init_cond_b = [3, -2, 19/2];
[t,y_2b] = ode45(f_2b,t_interval , init_cond_b);
subplot(1,2,2);
plot(t,y_2b(:,1),'r','linewidth',3);
title('Solution from 1.2b')
```

Then we convert the given initial value problem to a system of three equations of the first order. If we set $y_3(t) = 2\,x''(t) = 2\,y_2'(t)$, we get the system of three equations

$$\begin{aligned} y_1' &= y_2, & y_1(0) &= 3, \\ y_2' &= y_3, & y_2(0) &= -2, \\ y_3' &= \frac{31}{4}\,y_2 - \frac{15}{4}\,y_1; & 2\,y_3(0) &= 19. \end{aligned}$$

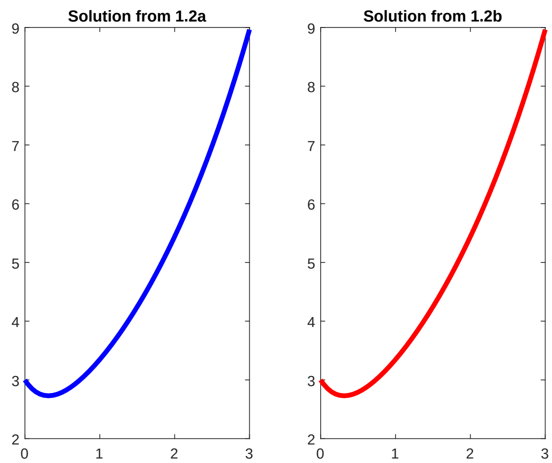

Figure 3: Solutions of Problem 2, parts (a) and (b), plotted with MATLAB.

This yields the initial value problem in vector equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1) & \\ 0 & 0 & 1 \\ -15/4 & 31/4 & 0 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}, \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 19/2 \end{bmatrix}$$

We invoke MATLAB and plot:

```
t_interval = [0,3];
init_cond_a = [3, -2]';
f_2a = @(t,y) [y(2); 3/2*y(1)-5/2*y(2)];
[t,y_2a] = ode45(f_2a,t_interval , init_cond_a);
%
f_2d = @(t,y) [y(2); y(3); -15/4*y(1) + 31/4*y(2)];
init_cond_b = [3, -2, 19/2];
[t,y_2d] = ode45(f_2d,t_interval , init_cond_b);
figure
plot(t, 2*y_2d(:,3) + 5*y_2a(:,2) - 3*y_2a(:,1),'linewidth',2)
title('1.2d:Difference 2y3 + (5x'''' - 3x'')')
```

1.3 (30 points) Consider a third order constant coefficient Genesio differential equation

$$x''' + a\,x'' + b\,x' + c\,x - x^2 + x^4 = 0,$$

where a, b, and c are some real constants.

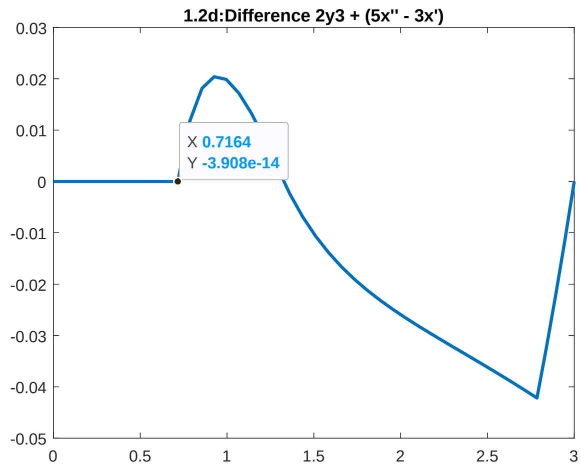

Figure 4: Difference of the second derivative and linear part, plotted with MATLAB.

(a) (10 points)
Convert this equation into a system of first order equations and rewrite it in vector form:

$$\dot{\mathbf{y}} = \mathbf{f}\left(\mathbf{y}\right),$$

clearly identifying the 3-by-1 vector $\mathbf{f}$ and 3-column vector $\mathbf{y}$.

(b) (10 points) Using an appropriate software package, solve numerically the Genesio equation for $a = c = 0$ subject to some initial conditions (not identically zero).

(c) (10 points) Repeat the previous problem for some small values of parameters a and c. Then plot its solution on the interval $[0, 3]$ along with the solution for $a = c = 0$.

Solution. Upon introducing three new dependent variables $y_1 = x$, $y_2 = \dot{x} = \dot{y}_1$, and $y_3 = \ddot{x} = \dot{y}_2$, we obtain

$$\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= y_3 \\
\dot{y}_3 &= -a\, y_3 - b\, y_2 - c\, y_1 + y_1^2 - y_1^4,
\end{aligned}$$

which we rewrite in matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ -a\, y_3 - b\, y_2 - c\, y_1 + y_1^2 - y_1^4 \end{bmatrix}.$$

Mathematica code:

```
sol3 = NDSolve[{x'''[t] + 1*x'[t] - (x[t])^2 + (x[t])^4 == 0, x[0] == 1, x'[0] == -0.
sol3a = NDSolve[{x'''[t] + 0.1*x''[t] + 1*x'[t] + 0.1*x[t] - (x[t])^2 + (x[t])^4 == (
Plot[{Evaluate[x[t] /. sol3], Evaluate[x[t] /. sol3a]}, {t, 0, 3}, PlotTheme -> "Mark
```

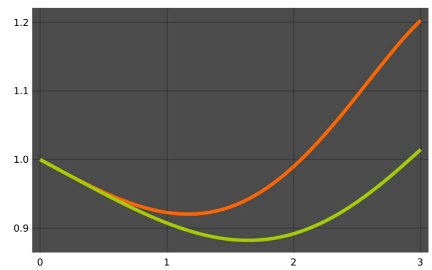

Figure 5: Solutions of Genesio's model with Mathematica.

We can do the same job with MATLAB:

```
f_3b = @(t,y) [y(2);y(3); y(1)^2 - y(1)^4-y(2)];
init_cond = [1, -.1, 0];
t_interval = [0,3];
[t,y_3b] = ode45(f_3b,t_interval , init_cond);
figure
plot(t, y_3b(:,1),'linewidth',2)
title('1.3b: Genesio equation with a = c = 0')
```

Then we repeat calculations with small values of parameters $a = c = 0.1$:

```
f_3b = @(t,y) [y(2);y(3); y(1)^2 - y(1)^4-y(2)];
init_cond = [1, -.1, 0];
t_interval = [0,3];
[t,y_3b] = ode45(f_3b,t_interval , init_cond);
figure
plot(t, y_3b(:,1))
title('1.3b: Genesio equation with a = c = 0')

%1.3c Repeat with small values of a and c and graph with 1.3b
figure
plot(t, y_3b(:,1),'b','linewidth',2)
hold on
f_3c = @(t,y) [y(2);y(3); y(1)^2 - y(1)^4-y(2)-.1*y(1) - .1*y(3)];
```

```
init_cond = [1, -.1, 0];
t_interval = [0,3];
[t,y_3c] = ode45(f_3c,t_interval , init_cond);
plot(t, y_3c(:,1),'r','linewidth',2)
title('1.3c: Genesio equation with a = c = 0.1')
```