AM 034 — Applied Mathematics - II

Brown University | **Fall 2021**

Homework, Set 3 | **Due October 06**

3.1 (20 points) For what vectors $\mathbf{b}$ does the linear system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a nontrivial solution?

$$(\mathbf{a})\quad \mathbf{A} = \begin{bmatrix} 2 & -1 & 8 \\ 1 & 3 & -3 \\ -1 & -2 & 1 \end{bmatrix}, \qquad (\mathbf{b})\quad \mathbf{A} = \begin{bmatrix} 2 & 3 & 9 \\ 1 & 2 & 5 \\ 1 & -2 & 1 \end{bmatrix}, \qquad (\mathbf{c})\quad \mathbf{A} = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix}.$$

3.2 (30 points) For each of the following matrices, find $\mathbf{U}(t) = e^{\mathbf{A}t}$ by performing the following steps:

(a) determine all eigenvalues and corresponding eigenvectors;

(b) Form a 3×3 matrix $\mathbf{S}$, where each column is an eigenvector;

(c) Calculate the diagonal matrix $e^{\mathbf{\Lambda}t}$ having on diagonal $e^{\lambda_j t}$, where λ_j $(j = 1, 2, 3)$ is an eigenvalue;

(d) Find $\mathbf{U}(t) = e^{\mathbf{A}t} = \mathbf{S}\, e^{\mathbf{\Lambda}t}\, \mathbf{S}^{-1}$.

$$(\mathbf{a})\quad \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \\ 4 & 2 & 1 \end{bmatrix}, \qquad (\mathbf{b})\quad \begin{bmatrix} 2 & -1 & 8 \\ 1 & 2 & -1 \\ 1 & 7 & -1 \end{bmatrix}, \qquad (\mathbf{c})\quad \begin{bmatrix} -10 & -6 & -3 \\ 12 & 8 & 3 \\ 12 & 6 & 5 \end{bmatrix}.$$

3.3 (40 points) For each matrix from the previous problem, perform the following steps.

(a) determine the minimal polynomial;

(b) find the Sylvester auxiliary matrix $\mathbf{Z}_k$ for each eigenvalue λ_k;

(c) verify the spectral decomposition $\mathbf{A} = \sum_k \lambda_k \mathbf{Z}_k$ for each matrix from the previous problem;

(d) construct the exponential matrix $\mathbf{U}(t) = e^{\mathbf{A}t}$ for each matrix from the previous problem;

(e) show that matrix $\mathbf{U}(t)$ is a (unique) solution of the matrix initial value problem

$$\dot{\mathbf{U}}(t) = \mathbf{A}\,\mathbf{U}(t), \qquad \mathbf{U}(0) = \mathbf{I}, \quad \text{the identity matrix.}$$

3.4 (10 points) Given the matrices $\mathbf{A}$ and $\mathbf{B}$ as shown below. Find eigenvalues of their products $\mathbf{A}\,\mathbf{B}$ and $\mathbf{B}\,\mathbf{A}$ and determine their algebraic and geometrical multiplicities.

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 2 & 1 \\ 7 & -3 & -4 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 6 & 3 \\ -1 & 0 \\ 0 & -4 \\ 2 & 1 \end{bmatrix}.$$