Week 9

Variation of Parameters

The method of undetermined coefficients is simple and has important applications, but it applies only to constant-coefficient equations with a special forcing function. Now we discuss the *general method* credited to **Lagrange**, which is known as the **method of variation of parameters**. This method can be applied to find a solution of a non-homogeneous linear differential equation with variable coefficients of any order. Suppose that we know a fundamental set of solutions for a corresponding homogeneous equation. With this in hand, we shall describe the Lagrange method for a non-homogeneous linear differential equation. Let us start, for simplicity, with the second order differential equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{9.1}$$

with arbitrary variable functions $p(x), q(x)$, and $f(x)$ that are continuous on some open interval (a, b). The continuity of $p(x)$ and $q(x)$ on the interval (a, b) implies that the associated homogeneous equation, $y'' + py' + qy = 0$, has the general solution

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x), \tag{9.2}$$

where C_1 and C_2 are arbitrary constants (or parameters) and $y_1(x)$, $y_2(x)$ is, of course, a known fundamental set of solutions. The method of variation of parameters involves "varying" the parameters C_1 and C_2, replacing them by functions $A(x)$ and $B(x)$ to be determined so that the resulting function

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) \tag{9.3}$$

is a particular solution of Eq. (9.1) on the interval (a, b). By differentiating Eq. (9.3) we obtain

$$y_p'(x) = A'(x)y_1(x) + A(x)y_1'(x) + B'(x)y_2(x) + B(x)y_2'(x).$$

Let us require that

$$A'(x)y_1(x) + B'(x)y_2(x) = 0. \tag{9.4}$$

Thus,

$$y_p'(x) = A(x)y_1'(x) + B(x)y_2'(x),$$

so the derivative, y_p', is obtained from Eq. (9.3) by differentiating only y_1 and y_2 but not $A(x)$ and $B(x)$. Hence,

$$y_p''(x) = A(x)y_1''(x) + A'(x)y_1'(x) + B(x)y_2''(x) + B'(x)y_2'(x).$$

Finally we substitute these expressions for y_p, y_p', and y_p'' into Eq. (9.1). After rearranging the terms in the resulting equation we find that

$$A(x)[y_1''(x) + p(x)y_1'(x) + q(x)y_1(x)] + B(x)[y_2''(x) + p(x)y_2'(x) + q(x)y_2(x)]$$

$$+A'(x)y_1'(x) + B'(x)y_2'(x) = f(x).$$

Since y_1 and y_2 are solutions of the homogeneous equation, namely,

$$y_j''(x) + p(x)y_j'(x) + q(x)y_j(x) = 0, \; j = 1, 2,$$

each of the expressions in brackets in the left-hand side is zero. Therefore, we have

$$A'(x)y_1'(x) + B'(x)y_2'(x) = f(x). \tag{9.5}$$

Equations (9.4) and (9.5) form a system of two linear algebraic equations for the unknown derivatives $A'(x)$ and $B'(x)$. The solution is obtained by Cramer's rule or Gaussian elimination. Thus, multiplying the equation (9.4) by $y_1'(x)$ and equation (9.5) by $-y_2(x)$ and adding, we get

$$A'(x)(y_1y_2' - y_2y_1') = -y_2(x)f(x).$$

The expression in parenthesis in the left-hand side is exactly the Wronskian of a fundamental set of solutions, that is, $W(y_1, y_2; x) = y_1y_2' - y_2y_1'$. Therefore, $A'(x)W(x) = -y_2(x)f(x)$. We then multiply Eq. (9.4) by $-y_1'(x)$ and Eq. (9.5) by $y_1(x)$ and add to obtain $B'(x)W(x) = y_1(x)f(x)$. The functions $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the homogeneous equation $y'' + py' + qy = 0$; therefore its Wronskian is not zero on the interval (a, b), where functions $p(x)$ and $q(x)$ are continuous. Now division by $W(x) \neq 0$ gives

$$A'(x) = -\frac{y_2(x)f(x)}{y_1y_2' - y_2y_1'}, \quad B'(x) = \frac{y_1(x)f(x)}{y_1y_2' - y_2y_1'}.$$

By integrating we have

$$A(x) = -\int \frac{y_2(x)f(x)}{W(y_1, y_2; x)}\,dx + C_1, \quad B(x) = \int \frac{y_1(x)f(x)}{W(y_1, y_2; x)}\,dx + C_2. \tag{9.6}$$

Substituting these integrals into Eq. (9.3), we obtain the general solution of the inhomogeneous equation:

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,dx + C_1y_1(x) + C_2y_2(x).$$

Since the linear combination $y_h(x) = C_1y_1(x) + C_2y_2(x)$ is a solution of the corresponding homogeneous equation, we can disregard it because our goal is to find a particular solution. Thus, a particular solution of non-homogeneous equation has the form

$$y_p(x) = \int G(x, \xi)\, f(\xi)\, d\xi, \tag{9.7}$$

where function $G(x, \xi)$, called the **Green function of the linear operator** $L = D^2 + p(x)D + q(x)$, is

$$G(x, \xi) = \frac{y_1(\xi)y_2(x) - y_2(\xi)y_1(x)}{y_1(\xi)y_2'(\xi) - y_2(\xi)y_1'(\xi)}. \tag{9.8}$$

The Green function depends only on the solutions y_1 and y_2 of the corresponding homogeneous equation $L[y] = 0$ and is independent of the forcing term. Therefore, $G(x, \xi)$ is completely determined by the linear differential operator L.