Week 11

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11 Power series solutions

If a real-valued function $f(x)$ has $N+1$ continuous derivatives on the interval $a \leqslant x \leqslant b$, then

$$f(x) = \sum_{n=0}^{N} c_n \left(x - x_0 \right)^n = \sum_{n=0}^{N} \frac{1}{n!}\, f^{(n)} \left(x_0 \right) \left(x - x_0 \right)^n + R_{N+1}(x), \qquad a < x < b,$$

where the remainder term is

$$R_{N+1} = \frac{f^{(N+1)}(\xi)}{(N+1)!} \left(x - x_0 \right)^{N+1}$$

for some (unknown ξ in the interval $a \leqslant x \leqslant b$. If we set $R_{N+1} = 0$, an N-th degree polynomial approximation to $f(x)$ is obtained; this is sometimes called the **Taylor polynomial** of degree N. If the remainder R_{N+1} tends to zero when $N \to \infty$, the the function has a power series representation

$$f(x) = \sum_{n\ge 0} c_n \left(x - x_0 \right)^n = \sum_{n\ge 0} \frac{1}{n!}\, f^{(n)} \left(x_0 \right) \left(x - x_0 \right)^n , \qquad a < x < b,$$

called the Taylor series for function f centered at x_0. If the center is zero, $x_0 = 0$, the series is usually referred to as the Maclaurin series. A function that has a convergent Taylor series in some neighborhood of the point x_0 is called holomorphic (or analytic). It turns out that if the power series converges at some point other than x_0, then it converges in a symmetric interval $|x - x_0| < R$, where R is called the radius of convergence. Moreover, a Taylor series for any holomorphic function is unique.

Since all coefficients in Taylor's series are evaluated at one single point—the center, this series provides information about the function locally. Therefore we expect that its truncated version—Taylor's polynomial—gives a good approximation within the interval of convergence. We try to use this property for constructing or approximating a solution to differential equation in the form of power series. To do this, we need to know one main property of Taylor's series: the derivative of any holomorphic function f exists and can be obtained by term-by=term differentiation:

$$f'(x) = \sum_{n\ge 1} n\, c_n \left(x - x_0 \right)^{n-1} = \sum_{k\ge 0} (k+1)\, c_{k+1} \left(x - x_0 \right)^k .$$

We will use another important property for multiplication of two power series:

$$\left(\sum_{n\ge 0} a_n \left(x - x_0 \right)^n \right) \left(\sum_{n\ge 0} b_n \left(x - x_0 \right)^n \right) = \sum_{n\ge 0} c_n \left(x - x_0 \right)^n ,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

is called the **convolution** of two series.

There are two main techniques to obtain power series representations of solutions to differential equations (ordinary and partial), and we consider them separately. To make exposition clear, we show how to apply them on typical examples.

11.1 Differential Transform Algorithm

We consider a typical first order differential equation subject to some initial condition

$$y' = y^2 - x^2, \qquad y(0) = 1/2.$$

Since the initial condition is set at the origin, it is natural to seek its solution as the Maclaurin power series:

$$y(x) = \sum_{n\ge 0} c_n x^n = \sum_{n\ge 0} \frac{1}{n!}\, y^{(n)}\,(0)\, x^n .$$

From the initial condition, we deduce that $c_0 = 1/2$. To find other coefficients, one needs to differentiate sequentially the above equation and set $x = 0$. This yields

$$\begin{aligned}
y'(0) &= \lim_{x\to 0} \left(y^2 - x^2 \right) = \lim_{x\to 0}\, y^2 = y(0)^2 = \frac{1}{4}, \\
y''(0) &= \lim_{x\to 0} \frac{textd}{\mathrm{d}x} \left(y^2 - x^2 \right) = \lim_{x\to 0} (2y\, y' - 2x) = 2 \left(\frac{1}{2} \right)^3 , \\
y'''(0) &= \lim_{x\to 0} \frac{textd^2}{\mathrm{d}x^2} \left(y^2 - x^2 \right) = \lim_{x\to 0} \left(6\, y^4 - 2 \right) = 3\, \frac{1}{8} - 2 = -\frac{13}{8}. \\
&\vdots
\end{aligned}$$

As you see, further calculations become quickly messy, and a computer algebra system is needed. This allows us to find some first terms in the solution power series:

$$y(x) = \frac{1}{2} + \frac{1}{4}\, x + \frac{1}{8}\, x^2 - \frac{13}{48}\, x^3 - \frac{5}{96}\, x^4 - \frac{11}{320}\, x^5 \cdots .$$

We check our answer with *Mathematica*:

```
AsymptoticDSolveValue[{y'[x] == y[x]*y[x] - x^2, y[0] == 1/2},
 y[x], {x, 0, 8}]
```

11.2 Derivation of Recurrences

An another approach to find the power series representation of a solution is to derive a recurrence for its coefficients. Substituting $y = \sum_{n\ge 0} c_n x^n$ into the differential equation yields

$$\sum_{n\ge 0} (n+1)\, c_{n+1} x^n = \sum_{n\ge 0} b_n x^n - x^2,$$

where

$$b_n = \sum_{k=0}^{n} c_k c_{n-k}$$

is a convolution of the series for $y(x)$ with itself. Comparing coefficients of like powers of x leads to the recurrence

$$\begin{aligned} c_1 &= b_0 = c_0^2 = \frac{1}{4}, \\ 2\,c_2 &= b_1 = 2\,c_0 c_1 = 2\,\frac{1}{8}, \\ 3\,c_3 &= b_2 - 1 = 2\,c_0 c_2 + c_1^2 - 1 = -\frac{13}{16}, \\ (n+1)\,c_{n+1} &= b_n = \sum_{k=0}^{n} c_k c_{n-k}, \qquad n = 3, 4, \ldots . \end{aligned}$$

Therefore, we get the full-history recurrence

$$c_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} c_k c_{n-k}, \qquad n = 3, 4, \ldots$$

11.3 Second Order ODEs

Consider the nonhomogeneous Chebyshev equation

$$\left(1 - x^2\right) y'' - x\, y' + 25\, y = x^2, \qquad y(0) = 1, \quad y'(0) = -1.$$

We seek its solution in the form of Maclaurin series

$$y(x) = \sum_{n\ge 0} c_n x^n = 1 - x + \sum_{n\ge n} c_n x^n$$

because the first two coefficients follow from the initial conditions. Differentiation of the series gives

$$\begin{aligned} x\, y' &= \sum_{n\ge 1} n\, c_n x^n, \\ y'' &= \sum_{n\ge 2} n(n-1)\, c_n x^{n-2} = \sum_{n\ge 0} (n+2)\,(n+1)\, c_{n+2} x^n, \end{aligned}$$

$$x^2 y'' = \sum_{n\geq 2} n(n-1)\, c_{n-2} x^n .$$

Substitution into the Chebyshev equation yields

$$\sum_{n\geq 0} (n+2)\,(n+1)\, c_{n+2} x^n - \sum_{n\geq 2} n(n-1)\, c_n x^n - \sum_{n\geq 1} n\, c_n x^n + 25 \sum_{n\geq 0} c_n x^n = x^2 .$$

Equating like powers of x, we obtain

$$\begin{aligned}
2\,c_2 + 25\,c_0 &= 0 & &\Longrightarrow & c_2 &= -\frac{25}{2}, \\
6\,c_3 - c_1 + 25\,c_1 &= 0 & &\Longrightarrow & c_3 &= -4, \\
12\,c_4 - 2\,c_2 - 2\,c_2 + 25\,c_2 &= 1 & &\Longrightarrow & c_4 &= -\frac{5}{3}, \\
(n+2)\,(n+1)\,c_{n+2} - n(n-1)\,c_n - n\,c_n + 25\,c_n &= 0, & & & n &= 3, 4, \ldots .
\end{aligned}$$

We check our answer with Maple:

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dsolve({diff(y(x),x,x)*(1-x^2) -x*diff(y(x),x) + 25*y(x)
= x^2, y(0) =1, D(y)(0) =-1},y(x),series)
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$$y(x) = 1 - x - \frac{25}{2}\, x^2 + 4\, x^3 + \frac{527}{24}\, x^4 - \frac{16}{5}\, x^5 + \cdots .$$