

# 1 The Laplace Transform

**Definition 1.** Let  $f$  be an arbitrary (complex valued or real valued) function, defined on the semi-infinite interval  $[0, \infty)$ ; then the integral

$$f^L(\lambda) = (\mathcal{L}f)(\lambda) = \int_0^{\infty} e^{-\lambda t} f(t) dt \quad (1)$$

is said to be the Laplace transform of  $f$ , if the integral (1) converges for some value  $\lambda = \lambda_0$ . Therefore the Laplace transform of a function (if it exists) depends on a parameter  $\lambda$  which could be either a real number or a complex number.

Saying that a function  $f(t)$  has a Laplace transform  $f^L(\lambda)$  means, that for some  $\lambda = \lambda_0$ , the limit

$$\lim_{N \rightarrow \infty} \int_0^N f(t) e^{-\lambda_0 t} dt = f^L(\lambda)$$

exists. The integral in the right-hand side of Eq.(1) is an integral over an unbounded interval. Such integrals are called *improper integrals* and they are defined as a limit of integrals over finite intervals. If such a limit does not exist the improper integral is said to diverge.

From the definition of the integral, it follows that if the Laplace transform exists for a particular function then it does not depend on the values of a function at a finite number of points. Namely, we can change the values of a function at a finite number of points and its Laplace transform will still be the same.

The parameter  $\lambda$  in the definition of the Laplace transform is not necessarily a positive or real number, but a complex number. Thus,  $\lambda = \alpha + i\beta$  where  $\alpha$  is the real part of  $\lambda$ , denoted by  $\alpha = \Re\lambda$  and  $\beta$  is an imaginary part of a complex number  $\lambda$ ,  $\beta = \Im\lambda$ . The set of all complex numbers is denoted as  $\mathbb{C}$  whereas the set of all real numbers is denoted as  $\mathbb{R}$ .

**Theorem 1.** The Laplace transform is a linear operator, that is

$$(\mathcal{L}Cf)(\lambda) = C(\mathcal{L}f)(\lambda), \text{ and } (\mathcal{L}(f+g))(\lambda) = (\mathcal{L}f)(\lambda) + (\mathcal{L}g)(\lambda),$$

where  $C$  is a constant and  $f$  and  $g$  are arbitrary functions for which the Laplace transforms exist.

**Theorem 2.** If a function  $f$  is absolutely integrable over finite intervals and the integral (1) converges for some complex number  $\lambda = \mu$ , then it converges in the half-space  $\Re\lambda > \Re\mu$ , i.e. in  $\{\lambda \in \mathbb{C} : \Re\lambda > \Re\mu\}$ .

There is some real value  $\sigma_c$  called the **abscissa of convergence** of the function  $f$  such that the integral (1) is convergent in the half-plane  $\Re\lambda > \sigma_c$  and divergent in the half-plane  $\Re\lambda < \sigma_c$ . We don't know precisely whether or not there are points of convergence on the line  $\Re\lambda = \sigma_c$  itself.

**Example 1.** Let  $p$  be any positive number (not necessarily an integer). Then the Laplace transform of the function  $f(t) = t^p$ ,  $t > 0$  is

$$\begin{aligned} (\mathcal{L}t^p)(\lambda) &= \int_0^{\infty} e^{-\lambda t} t^p dt = \int_0^{\infty} e^{-\lambda t} (\lambda t)^p \lambda^{-p} dt \\ &= \lambda^{-p-1} \int_0^{\infty} e^{-\tau} \tau^p d\tau = \frac{\Gamma(p+1)}{\lambda^{p+1}}, \end{aligned}$$

where

$$\Gamma(\nu) = \int_0^{\infty} e^{-\tau} \tau^{\nu-1} d\tau \quad (2)$$

is the **Gamma function of Euler**. This improper integral converges for  $\nu > 0$  and by integrating by parts we obtain

$$\Gamma(\nu+1) = \nu\Gamma(\nu). \quad (3)$$

Indeed, for  $\nu > 0$  we have

$$\begin{aligned} \Gamma(\nu+1) &= \int_0^{\infty} e^{-\tau} \tau^{\nu} d\tau = - \int_0^{\infty} \tau^{\nu} de^{-\tau} \\ &= -\tau^{\nu} e^{-\tau} \Big|_{\tau=0}^{\tau=\infty} + \nu \int_0^{\infty} e^{-\tau} \tau^{\nu-1} d\tau = \nu\Gamma(\nu). \end{aligned}$$

The most remarkable property of the  $\Gamma$ -function is obtained when we set  $\nu = n$ , an integer. The comparison with the result of the previous example yields

$$\Gamma(n+1) = n!, \quad n = 0, 1, 2, \dots$$

**Definition 2.** A function  $f$  is said to be **piecewise continuous** on a finite interval  $[a, b]$  if this interval can be subdivided into finitely many intervals so that  $f(t)$  is continuous on each subinterval and approaches a finite limit at the end points of each subinterval from the interior. That is, there are finite number of points  $\{\alpha_j\}$ ,  $j = 1, 2, \dots, N$  where a function  $f$  has a jump discontinuity when both

$$\lim_{\substack{h \rightarrow 0 \\ h \geq 0}} f(\alpha_j + h) = f(\alpha_j + 0) \quad \text{and} \quad \lim_{\substack{h \rightarrow 0 \\ h \geq 0}} f(\alpha_j - h) = f(\alpha_j - 0)$$

exist but are different.

Note that an infinite number of discontinuities is allowed, as long as just a finite number occur on a finite interval. However, all these jumps must be finite.

Remember that for a continuous function  $f$  we have  $f(t) = f(t+0) = f(t-0)$ . If, at some point  $t = t_0$  this is not valid, then a function is discontinuous at  $t = t_0$ . In other words, the finite discontinuity occurs if the left hand side and the right hand side limits are finite and are not equal.

**Definition 3.** The Heaviside function  $H(t)$  is the unit step function, equal to zero for  $t$  negative and unity for  $t$  positive, with  $H(0) = 1/2$ , i.e.

$$H(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0 \end{cases} \quad (4)$$

**Remark.** Of course, we can define the value of the Heaviside function at the point  $t = 0$  whatever we want. As we will see in the next example, it will not effect the value of its Laplace transformation. That is why we can change the value at a finite number of points of any function and it will not change the value of the corresponding Laplace transform. But if we wish to restore a function from it's Laplace transform value we will get a function that possesses the same property as the Heaviside function. Namely, a value of the function at any point will be equal to the mean of it's right hand side and left hand side limit values:

$$f(t) = \frac{1}{2}[f(t+0) + f(t-0)]. \quad \square$$

**Definition 4.** A function  $f(t)$ ,  $t \in [0, \infty)$  is said to be a **function-original** if it has on every finite interval only a finite number of points of discontinuity and

$$|f(t)| < M e^{ct} \quad (t > T) \quad (5)$$

for some values of  $c, M$  and  $T$ , which may be very large. Moreover we assume that at points of discontinuity the value of a function-original is equal to the corresponding mean value, thus

$$f(t_0) = \frac{1}{2}[f(t_0+0) + f(t_0-0)] = \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon \geq 0}} \frac{f(t_0 + \varepsilon) + f(t_0 - \varepsilon)}{2}. \quad (6)$$

The Laplace transform of such a function is called the **image**.

**Definition 5.** We say that a function  $f$  is of exponential order if for some values  $c, M$ , and  $T$  (5) holds. We abbreviate this as  $f = O(e^{ct})$ . A function  $f$  is said to be of exponential order  $\alpha$ , or  $eo(\alpha)$  for abbreviation, if  $f = O(e^{ct})$  for any real number  $c > \alpha$ , but not when  $c < \alpha$ .

**Definition 6.** The integral (1) is said to be absolutely convergent, if the integral

$$\int_0^{\infty} e^{-\Re\lambda t} |f(t)| dt \quad (7)$$

converges. The greatest lower bound  $\sigma_a$  of such numbers  $\Re\lambda$  for which the integral (7) converges is called the abscissa of absolute convergence.

**Theorem 3.** If  $|f(t)| \leq C$  for  $t \geq T$ , then the Laplace transform (1) converges absolutely for any  $\lambda_0$  with  $\Re\lambda_0 > 0$ . In particular, the Laplace transform exists for any positive (real)  $\lambda$ .

**Theorem 4.** The integral (1) converges for any function-original. Moreover, if a function  $f$  is of exponential order  $\alpha$ , then the integral (1) absolutely converges for  $\Re\lambda > \alpha$ . Furthermore, if  $f$  and  $g$  are piecewise continuous functions whose Laplace transforms exist and satisfy  $(\mathcal{L}f) = (\mathcal{L}g)$ , then  $f = g$  at their points of continuity. Thus, if  $F(\lambda)$  has a continuous inverse  $f$ , then  $f$  is unique.

#	Function-original	Its Laplace Transform
1.	$H(t)$	$\frac{1}{\lambda}$
2.	$H(t - a)$	$\frac{1}{\lambda} e^{-a\lambda}$
3.	$t$	$\frac{1}{\lambda^2}$
4.	$t^n, n = 1, 2, \dots$	$\frac{n!}{\lambda^{n+1}}$
5.	$t^p$	$\frac{\Gamma(p+1)}{\lambda^{p+1}}$
6.	$e^{\alpha t}$	$\frac{1}{\lambda - \alpha}, \Re \lambda > \Re \alpha$
7.	$t^n e^{\alpha t}, n = 1, 2, \dots$	$\frac{n!}{(\lambda - \alpha)^{n+1}}, \Re \lambda > \Re \alpha$
8.	$\sin \alpha t$	$\frac{\alpha}{\lambda^2 + \alpha^2}, \Re \lambda > 0$
9.	$\cos \alpha t$	$\frac{\lambda}{\lambda^2 + \alpha^2}, \Re \lambda > 0$
10.	$e^{\alpha t} \sin \beta t$	$\frac{\beta}{(\lambda - \alpha)^2 + \beta^2}, \Re \lambda > \Re \alpha$
11.	$e^{\alpha t} \cos \beta t$	$\frac{\lambda - \alpha}{(\lambda - \alpha)^2 + \beta^2}, \Re \lambda > \Re \alpha$
12.	$\sinh \beta t$	$\frac{\beta}{\lambda^2 - \beta^2}, \Re \lambda > \Re \beta$
13.	$\cosh \beta t$	$\frac{\lambda}{\lambda^2 - \beta^2}, \Re \lambda > \Re \beta$
14.	$t \sin \beta t$	$\frac{2\beta\lambda}{(\lambda^2 + \beta^2)^2}, \Re \lambda > 0$
15.	$t \cos \beta t$	$\frac{\lambda^2 - \beta^2}{(\lambda^2 + \beta^2)^2}, \Re \lambda > 0$
16.	$e^{\alpha t} - e^{\beta t}$	$\frac{\alpha - \beta}{(\lambda - \alpha)(\lambda - \beta)}, \Re \lambda > \Re \alpha, \Re \beta$
17.	$e^{\alpha t} \left[ \cos \beta t + \frac{\alpha}{\beta} \sin \beta t \right]$	$\frac{\lambda}{(\lambda - \alpha)^2 + \beta^2}, \Re \lambda > \Re \alpha$
18.	$\frac{\sin \beta t}{2\beta^3} - \frac{t \cos \beta t}{2\beta^2}$	$\frac{1}{(\lambda^2 + \beta^2)^2}, \Re \lambda > 0$
19.	$\frac{t \sin \beta t}{2\beta}$	$\frac{\lambda}{(\lambda^2 + \beta^2)^2}, \Re \lambda > 0$
20.	$e^{\alpha t} \sinh \beta t$	$\frac{\beta}{(\lambda - \alpha)^2 - \beta^2}, \Re \lambda > \Re(\alpha \pm \beta)$
21.	$e^{\alpha t} \cosh \beta t$	$\frac{\lambda - \alpha}{(\lambda - \alpha)^2 - \beta^2}, \Re \lambda > \Re(\alpha \pm \beta)$

Table 1: A Table of Elementary Laplace Transforms. Note: Each function in the left column is zero for negative  $t$ ; that is, they must be multiplied by the Heaviside function  $H(t)$ .

## 2 Properties of the Laplace Transform

The success of transformation techniques in solving initial value problems and other applications hinges on their *operational properties*. Rules that govern how operations in the time domain translate to operations in the image domain are called *operational laws* or *rules*. In this section we present the basic 6 rules that are useful in applications of the Laplace Transformation to differential equations. The justification of these laws involve technical details that are beyond the scope of the text and therefore, is omitted. So we simply point to [1], [2]. We start with the following

**Definition 7.** *The convolution of two functions  $f$  and  $g$ , defined on the half-line  $[0, \infty)$ , is the integral*

$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau) d\tau = (g * f)(t).$$

It is easy to verify that the convolution of two constants is

$$1 * 1 = \int_0^t d\tau = t.$$

Many examples of convolutions the reader will find later in the following sections. Now we list the properties of the Laplace transform.

### 1° The differential rule

$$\mathcal{L}[f^{(n)}(t)](\lambda) = \lambda^n \mathcal{L}f(\lambda) - \sum_{k=1}^n \lambda^{n-k} f^{(k-1)}(+0). \quad (8)$$

Integration by parts gives us the equality (8). In particular,

$$\mathcal{L}[f'(t)](\lambda) = \lambda f^L(\lambda) - f(0). \quad (9)$$

$$\mathcal{L}[f''(t)](\lambda) = \lambda^2 f^L(\lambda) - \lambda f(0) - f'(0). \quad (10)$$

### 2° The convolution rule

The Laplace transform of the convolution of two functions is equal to the product of its images:

$$\mathcal{L}(f * g)(\lambda) = f^L(\lambda)g^L(\lambda). \quad (11)$$

### 3° The similarity rule

$$\mathcal{L}[f(at)](\lambda) = \frac{1}{a} f^L\left(\frac{\lambda}{a}\right), \quad \Re\lambda > a\sigma_c, \quad (12)$$

if  $a$  is a positive number.

### 4° The shift rule

If we know  $G(\lambda)$ , which is the Laplace transform of  $g(t)$ , then the retarded function  $f(t) = g(t-a)H(t-a)$  has the Laplace transform  $G(\lambda)e^{-\lambda a}$ , namely,

$$\mathcal{L}[H(t-a)g(t-a)](\lambda) = e^{-a\lambda} g^L(\lambda), \quad a > 0, \quad (13)$$

Similarly

$$\mathcal{L}[f(t+a)](\lambda) = e^{a\lambda} \left\{ f^L(\lambda) - \int_0^a e^{-\lambda t} f(t) dt \right\}, \quad a > 0. \quad (14)$$

where  $H$  is the Heaviside function.

### 5° The attenuation rule

$$\mathcal{L}[e^{-at} f(t)](\lambda) = f^L(\lambda + a). \quad (15)$$

### 6° The integration rule

$$\mathcal{L}[t^{n-1} * f(t)](\lambda) = \mathcal{L} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau = \frac{(n-1)!}{\lambda^n} f^L(\lambda), \quad n = 1, 2, \dots \quad (16)$$

If  $n = 1$ , then

$$\frac{1}{\lambda} f^L(\lambda) = \mathcal{L} \int_0^t f(\tau) d\tau. \quad (17)$$

**Remark.** We can unite (12) and (15) so that

$$\mathcal{L} \left[ \frac{1}{a} e^{-bt/a} f \left( \frac{t}{a} \right) \right] (\lambda) = f^L(a\lambda + b). \quad (18)$$

## 3 The Inverse Laplace Transform

We employ the symbol  $\mathcal{L}^{-1}[F(\lambda)]$ , corresponding to the direct Laplace transform defined by Eq.(1.1), to denote a function  $f(t)$  whose Laplace transform is  $F(\lambda)$ . Thus, we have the Laplace pair

$$F(\lambda) = (\mathcal{L}f(t))(\lambda), \quad f(t) = \mathcal{L}^{-1}[F(\lambda)](t).$$

It has already been demonstrated that the Laplace transform  $f^L(\lambda)$  of a given function  $f(t)$  can be calculated by direct integration. The inverse Laplace transform is more complicated. However, it is very important because the solution of practical problems usually provides a known  $F(\lambda)$  from which the function  $f(t)$  must be found such that  $f^L(\lambda) = F(\lambda)$ . Thomas John l'Anson Bromwich (1875 – 1929) answered the question of how to find this function,  $f(t)$ , which is the inverse Laplace transform of a given function  $F(\lambda)$ , in 1916. He expressed the inverse Laplace transform as the contour integral

$$\frac{1}{2}[f(t+0) + f(-0)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^L(\lambda) e^{\lambda t} d\lambda, \quad (19)$$

where  $c$  is any number greater than the abscissa of convergence for  $f^L(\lambda)$  and the integration is defined in the sense of the Cauchy principle value.

**Remark.** From this formula (19), it follows that the inverse Laplace transform restores a function-original from its image in such a way that the value of a function-original at any point is equal to a mean of its right-hand side limit value and its left-hand side limit value. If a function is continuous at a point then its value at this point coincides with its mean value.  $\square$

In this section we will not use Eq. (19) as it is very complicated. Instead, we consider three methods to find the inverse Laplace transform: *Partial Fraction Decomposition*, the *Convolution Theorem*, and the *Cauchy Residue Theorem*. We will restrict ourselves to finding the inverse Laplace transform of rational functions or their products on exponentials, that is,

$$F(\lambda) = \frac{P(\lambda)}{Q(\lambda)} \quad \text{or} \quad F_a(\lambda) = \frac{P(\lambda)}{Q(\lambda)} e^{-\alpha\lambda},$$

where  $P(\lambda)$  and  $Q(\lambda)$  are polynomials<sup>1</sup>. This case is one of the most important in applications of the Laplace transform to differential equations with constant coefficients. In this section we only consider cases in which the degree of the denominator is larger than the degree of the numerator.

The case of the product of a rational function and an exponential can be easily reduced to the case without the exponential multiplier by the shift rule (13). In fact, suppose we know

$$f(t) = \mathcal{L}^{-1}[F(\lambda)](t) = \mathcal{L}^{-1} \left[ \frac{P(\lambda)}{Q(\lambda)} \right] (t),$$

the original of a rational function  $F(\lambda)$ . Then according to (13) we have

$$H(t-a)f(t-a) = \mathcal{L}^{-1}[F(\lambda) e^{-a\lambda}].$$

<sup>1</sup>The result is valid for the case when  $Q$  is an entire function, that is,  $Q(\lambda)$  is represented by a series which converges everywhere except infinity.

### 3.1 Partial Fraction Decomposition

The fraction

$$F(\lambda) = \frac{P(\lambda)}{Q(\lambda)}$$

can be easily expanded into partial fractions, that is,  $P/Q$  can be represented as a linear combination of simple rational functions of the form  $1/(\lambda - \alpha)$ ,  $1/(\lambda - \alpha)^2$ , and so forth. To do this, it is first necessary to find all nulls of the denominator  $Q(\lambda)$  or, equivalently, to find all roots of the equation

$$Q(\lambda) = 0. \quad (20)$$

Then  $Q(\lambda)$  can be factored as

$$Q(\lambda) = c_0(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k},$$

where  $\lambda_1, \lambda_2, \dots, \lambda_k$  are the distinct roots of Eq. (20) and  $m_1, m_2, \dots, m_k$  are their respective **multiplicities**. A root of the Eq. (20) is called **simple** if its multiplicity equals 1. A root which appears twice is often called a **double root**. Recall that a polynomial of degree  $n$  has  $n$  roots, counting multiplicities, so  $m_1 + m_2 + \cdots + m_k = n$ . Thus if the equation (20) has a simple real root  $\lambda = \lambda_0$  then the polynomial  $Q(\lambda)$  has a factor  $\lambda - \lambda_0$ . To this factor in  $F = P/Q$  corresponds the partial fraction decomposition of the form

$$\frac{A}{\lambda - \lambda_0},$$

where  $A$  is a constant to be found. The inverse Laplace transform of this fraction is (see Table 1, formula 6)

$$\mathcal{L}^{-1} \left[ \frac{A}{\lambda - \lambda_0} \right] = A e^{\lambda_0 t} H(t),$$

where  $H$  is the Heaviside function (4).

The attenuation rule (15) gives us the clue about how get rid of  $\lambda_0$  in the denominator. Thus, using formula 1 from Table 1, yields

$$\mathcal{L}^{-1} \left[ \frac{A}{\lambda} \right] = AH(t).$$

Therefore,  $\mathcal{L} [e^{\lambda_0 t} AH(t)] = (\lambda - \lambda_0)^{-1}$ .

If a polynomial  $Q(\lambda)$  has a repeated factor  $(\lambda - \lambda_0)^m$ , that is, if Eq. (20) has a root  $\lambda_0$  with multiplicity  $m$ , then the partial fraction decomposition of  $F = P/Q$  contains a sum of  $m$  fractions

$$\frac{A_m}{(\lambda - \lambda_0)^m} + \frac{A_{m-1}}{(\lambda - \lambda_0)^{m-1}} + \cdots + \frac{A_1}{\lambda - \lambda_0}.$$

The inverse Laplace transform of each term is (see Table 1, formula 7):

$$\mathcal{L}^{-1} \left[ \frac{A_m}{(\lambda - \lambda_0)^m} \right] = A_m \frac{t^{m-1}}{(m-1)!} e^{\lambda_0 t} H(t).$$

Suppose a polynomial  $Q(\lambda)$  has an unrepeated complex factor  $(\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)$ , where  $\lambda_0 = \alpha + i\beta$ , and  $\bar{\lambda}_0 = \alpha - i\beta$  is the complex conjugate of  $\lambda_0$ . The coefficients of  $Q(\lambda)$  are real, complex roots occur in conjugate pairs. The pair of conjugate roots of Eq. (20) corresponding to this factor gives rise to the term

$$\frac{A\lambda + B}{(\lambda - \alpha)^2 + \beta^2}$$

in the partial fraction decomposition, since

$$(\lambda - \lambda_0)(\lambda - \bar{\lambda}_0) = (\lambda - \alpha)^2 + \beta^2.$$

The expansion of the fraction  $F = P/Q$  can be rewritten as

$$\frac{A(\lambda - \alpha) + \alpha A + B}{(\lambda - \alpha)^2 + \beta^2}.$$

From Table 1 formulas 9 and 10 and the shift rule (13) we obtain the inverse transform

$$\mathcal{L}^{-1} \left\{ \frac{A\lambda + B}{(\lambda - \alpha)^2 + \beta^2} \right\} = e^{\alpha t} \left[ A \cos \beta t + \frac{\alpha A + B}{\beta} \sin \beta t \right] H(t). \quad (21)$$

If the polynomial  $Q(\lambda)$  has the repeated complex factor  $[(\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)]^2$  then the sum of the form

$$\frac{A\lambda + B}{[(\lambda - \alpha)^2 + \beta^2]^2} + \frac{C\lambda + D}{(\lambda - \alpha)^2 + \beta^2}.$$

corresponds to this factor in partial fraction decomposition of  $F = P/Q$ . The last fraction is as in Eq. (21). To find the inverse Laplace transform of the first factor we can use formulas 17 and 18 from Table 1 and the shift rule (13). This leads us to

$$\mathcal{L}^{-1} \left\{ \frac{A\lambda + B}{[(\lambda - \alpha)^2 + \beta^2]^2} \right\} = e^{\alpha t} \left[ \frac{A}{2\beta} t \sin \beta t + \frac{\alpha A + B}{2\beta^2} (\sin \beta t - \beta t \cos \beta t) \right] H(t).$$

### 3.2 Convolution Theorem

Let the given function  $F(\lambda)$  be represented as a product of two other functions:  $F(\lambda) = F_1(\lambda) \cdot F_2(\lambda)$ . Assume that we know the inverse Laplace transforms  $f_1(t)$  and  $f_2(t)$  of these functions  $F_1$  and  $F_2$ . Then the inverse Laplace transform can be defined according to the convolution rule (11) to obtain

$$\mathcal{L}^{-1} \{F(\lambda)\} (t) = (f_1 * f_2)(t) \equiv \int_0^t f_1(\tau) f_2(t - \tau) d\tau = \int_0^t f_1(t - \tau) f_2(\tau) d\tau.$$

It turns out that one can calculate the inverse of such a product in terms of the known inverses, with the intervention of an integral.

### 3.3 Residue Method

Suppose a function  $F(\lambda) = P(\lambda)/Q(\lambda)$  is a fraction of two polynomials (or entire functions). We denote by  $\lambda_j$ ,  $j = 1, 2, \dots, N$  all nulls<sup>2</sup> of the denominator  $Q(\lambda)$ . Then the inverse Laplace transform of a function  $F$  can be found as

$$f(t) = \mathcal{L}^{-1} \{F(\lambda)\} = \sum_{j=1}^N \operatorname{Res}_{\lambda_j} F(\lambda) e^{\lambda t}, \quad (22)$$

where the sum covers all zeros of the equation (20) and residues  $\operatorname{Res}_{\lambda_j} F(\lambda) e^{\lambda t}$  are evaluated as follows.

If  $\lambda_j$  is a simple root of Eq. (20) then

$$\operatorname{Res}_{\lambda_j} F(\lambda) e^{\lambda t} = \frac{P(\lambda_j)}{Q'(\lambda_j)} e^{\lambda_j t}. \quad (23)$$

If  $\lambda_j$  is a double root of Eq. (20) then

$$\operatorname{Res}_{\lambda_j} F(\lambda) e^{\lambda t} = \lim_{\lambda \rightarrow \lambda_j} \frac{d}{d\lambda} \{(\lambda - \lambda_j)^2 F(\lambda) e^{\lambda t}\}. \quad (24)$$

In general, when  $\lambda_j$  is a  $n$ -fold root of Eq. (20) then

$$\operatorname{Res}_{\lambda_j} F(\lambda) e^{\lambda t} = \lim_{\lambda \rightarrow \lambda_j} \frac{1}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} \{(\lambda - \lambda_j)^n F(\lambda) e^{\lambda t}\}. \quad \square \quad (25)$$

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<sup>2</sup> $N = \infty$  if  $Q(\lambda)$  is an entire function.