The Laplace Transform and Initial Value Problems

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Chapter 1

The Laplace transform

In section 1.1, we introduce the Laplace transform. In section 1.2 and section 1.3, we discuss step functions and convolutions, two concepts that will be important later. In section 1.4, we discuss useful properties of the Laplace transform. In section 1.5 we do numerous examples of finding Laplace transforms.

1.1 Definition of the Laplace transform

In this section, we introduce the Laplace transform.

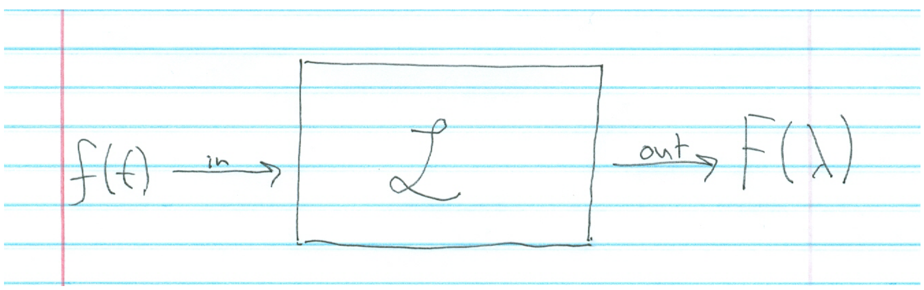
Definition 1.1 (Laplace transform of a function)**.** Let $f(t)$ be a piecewise continuous function defined for $t \geq 0$. Then the *Laplace transform* of f, denoted $\mathcal{L}\left[f(t)\right](\lambda)$, is defined as:

$$\mathcal{L}\left[f(t)\right](\lambda) := \int_0^\infty e^{-\lambda t} f(t)\,\mathrm{d}t = \lim_{b\to\infty}\left(\int_0^b e^{-\lambda t} f(t)\,\mathrm{d}t\right)$$

Alternative notations for the Laplace transform of $f(t)$ are $\mathcal{L}[f]$, $F(\lambda)$, and $f^L(\lambda)$.

You can think of the Laplace transform as some kind of abstract "machine." It takes in a function $f(t)$ and spits out a new function $F(\lambda)$.

Figure 1.1: The Laplace transform as a metaphorical "machine."



Example 1.2 (Laplace transform from definition)**.** Find the Laplace transform of $f(t) = 1$.

Solution: We directly apply the definition:

$$\mathcal{L}[1](\lambda) = \lim_{b\to\infty}\left(\int_0^b e^{-\lambda t}\,\mathrm{d}t\right) = \lim_{b\to\infty}\left(\left[-\frac{1}{\lambda}e^{-\lambda t}\right]_0^b\right)$$
$$= \lim_{b\to\infty}\left(-\frac{1}{\lambda}e^{-\lambda b} + \frac{1}{\lambda}e^{-\lambda 0}\right) = 0 + \frac{1}{\lambda} = \boxed{\frac{1}{\lambda}}$$

□

1.2 Step functions

In this section, we introduce the step function and demonstrate its utility.

Definition 1.3 (Heaviside step function)**.** The *Heaviside step function* (or simply *step function*), denoted $H(t)$, is defined by:

$$H(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}, & t = 0 \\ 1, & t > 0 \end{cases}$$

Step functions are nice because they allow us to write piecewise functions in a more compact way. Here is a formula for rewriting a piecewise function in terms of step functions.

Formula 1.4 (Piecewise functions in terms of step functions)**.** Let $g(t)$ be a function of the form:

$$g(t) = \begin{cases} h_1(t) & 0 \leq t < a_1 \\ h_2(t) & a_1 \leq t < a_2 \\ & \vdots \\ h_{m-1}(t) & a_{m-2} \leq t < a_{m-1} \\ h_m(t) & a_{m-1} \leq t \end{cases}$$

Then we can rewrite $h(t)$ according to the following equation:

$$\begin{aligned} g(t) = h_1(t) \cdot [H(t) - H(t-a_1)] + h_2(t) \cdot [H(t-a_1) - H(t-a_2)] \\ + \cdots + h_{m-1}(t) \cdot [H(t-a_{m-2}) - H(t-a_{m-1})] + h_m(t) \cdot [H(t-a_{m-1})] \end{aligned}$$

Here's an example to illustrate the use of this formula.

Example 1.5 (Rewriting piecewise functions)**.** Suppose $g(t)$ is given by:

$$g(t) = \begin{cases} 2 & 0 \leq t < 4 \\ 5 & a_1 \leq t < 7 \\ -1 & a_{m-2} \leq t < 9 \\ 1 & 9 \leq t \end{cases}$$

Write $g(t)$ in terms of step functions.

Solution: We apply Formula 1.4 and simplify:

$$\begin{aligned} g(t) &= 2 \cdot [H(t) - H(t-4)] + 5 \cdot [H(t-4) - H(t-7)] \\ &\quad - 1 \cdot [H(t-7) - H(t-9)] + 1 \cdot [H(t-9)] \\ &= 2H(t) - 2H(t-4) + 5H(t-4) - 5H(t-7) \\ &\quad - H(t-7) + H(t-9) + H(t-9) \\ &= 2H(t) + 3H(t-4) - 6H(t-7) + 2H(t-9) \end{aligned}$$

So we have $\boxed{g(t) = 2H(t) + 3H(t-4) - 6H(t-7) + 2H(t-9)}$. □

1.3 Convolutions

In this section, we introduce the convolution operation.

Definition 1.6 (Convolution of two functions)**.** Let $f(t)$ and $g(t)$ be defined for $t \geq 0$. The *convolution* (also referred to as the *convolution product*) of f and g, denoted $f * g$, is defined by:

$$(f * g)(t) := \int_0^t f(t-x)g(x)\,\mathrm{d}x$$

Note that the convolution is a *commutative* operation, i.e.

$$f * g = g * f$$

Example 1.7 (Computing a convolution product)**.** Let $f(t) = e^{3t}$ and let $g(t) = e^{7t}$. What is the convolution $f * g$?

Solution:

$$(f * g)(t) = \int_0^t f(t-x)g(x)\,\mathrm{d}x = \int_0^t e^{3(t-x)}e^{7x}\,\mathrm{d}x = \int_0^t e^{3t-3x}e^{7x}\,\mathrm{d}x$$

$$= \int_0^t e^{3t}e^{-3x+7x}\,\mathrm{d}x = e^{3t}\int_0^t e^{4x}\,\mathrm{d}x = e^{3t}\left[\frac{1}{4}e^{4x}\right]_0^t = e^{3t}\left(\frac{1}{4}e^{4t} - \frac{1}{4}e^{0}\right)$$

$$= \boxed{\frac{1}{4}e^{7t} - \frac{1}{4}e^{3t}}$$

□

1.4 Properties of the Laplace transform

In this section, we present some important properties of the Laplace transform.

1.4.1 Linearity

The Laplace transform is a *linear* operator. That is, if a, b are constants, then:

$$\mathcal{L}\left[a \cdot f(t) + b \cdot g(t)\right](\lambda) = a \cdot \mathcal{L}\left[f(t)\right](\lambda) + b \cdot \mathcal{L}\left[g(t)\right](\lambda) = a \cdot F(\lambda) + b \cdot G(\lambda)$$

1.4.2 Convolution rule

The Laplace transform of the convolution of f and g is equal to the product of the Laplace transformations of f and g, i.e.

$$\mathcal{L}\left[f * g\right](\lambda) = F(\lambda) \cdot G(\lambda)$$

In other words, the Laplace transform "turns convolution into multiplication."

1.4.3 Derivative rule

First derivative:

$$\mathcal{L}\left[f'(t)\right](\lambda) = \lambda F(\lambda) - f(0)$$

Second derivative:

$$\mathcal{L}\left[f''(t)\right](\lambda) = \lambda^2 F(\lambda) - \lambda f(0) - f'(0)$$

n-th derivative:

$$\mathcal{L}\left[f^{(n)}(t)\right](\lambda) = \lambda^n F(\lambda) - \sum_{k=1}^{n} \lambda^{n-k} f^{(k-1)}(0)$$

1.4.4 Similarity rule

Let $a > 0$ be a constant. Then:

$$\mathcal{L}\left[f(at)\right](\lambda) = \frac{1}{a}F\left(\frac{\lambda}{a}\right)$$

1.4.5 Shift rule

Let $a > 0$ be a constant. Then:

$$\mathcal{L}\left[H(t-a)g(t-a)\right](\lambda) = G(\lambda)e^{-a\lambda}$$

(Recall that $H(t)$ denotes the Heaviside step function.)

1.4.6 Attenuation rule

Let a be a constant. Then:

$$\mathcal{L}\left[f(t)e^{-at}\right](\lambda) = F(\lambda + a)$$

1.4.7 Rule for multiplication by t^n

Let n be a nonnegative integer (i.e. $n = 0, 1, 2, \ldots$). Then:

$$\mathcal{L}\left[t^n f(t)\right](\lambda) = (-1)^n \cdot \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[F(\lambda)\right]$$

1.4.8 Rule for division by t

$$\mathcal{L}\left[\frac{f(t)}{t}\right](\lambda) = \int_{\lambda}^{\infty} F(\sigma)\,\mathrm{d}\sigma$$

1.4.9 Rule for periodic functions

If $f(t+\omega) = f(t)$, then:

$$F(\lambda) = \frac{1}{1 - e^{-\omega\lambda}} \int_0^{\omega} e^{-\lambda t} f(t)\,\mathrm{d}t$$

1.4.10 Rule for anti-periodic functions

If $f(t+\omega) = -f(t)$, then:

$$F(\lambda) = \frac{1}{1 + e^{-\omega\lambda}} \int_0^{\omega} e^{-\lambda t} f(t)\,\mathrm{d}t$$

1.5 Examples

In this section, we provide some examples of how the properties discussed in the previous section can be used to compute Laplace transforms without actually evaluating any complicated integrals. Observe how the later examples take advantage of results derived in earlier examples.

Example 1.8 (Derivative rule)**.** Find $\mathcal{L}\left[\cos(t)\right](\lambda)$.

Solution: We will exploit the fact that differentiating cosine twice gives a negated cosine. We have:

$$\begin{aligned} f(t) &= \cos(t) \\ f'(t) &= -\sin(t) \\ f''(t) &= -\cos(t) = -f(t) \end{aligned}$$

Observe that by the linearity of the Laplace transform, we have that:

$$\begin{aligned} \mathcal{L}\left[f''(t)\right](\lambda) = \mathcal{L}\left[-f(t)\right](\lambda) = -\mathcal{L}\left[f(t)\right](\lambda) = -F(\lambda) \\ \implies F(\lambda) = -\mathcal{L}\left[f''(t)\right](\lambda) \end{aligned}$$

Applying the derivative rule:

$$\begin{aligned} F(\lambda) &= -\mathcal{L}\left[f''(t)\right](\lambda) = -\Big(\lambda^2 F(\lambda) - \lambda f(0) - f'(0)\Big) = -\Big(\lambda^2 F(\lambda) - \lambda\cos(0) - \sin(0)\Big) \\ &= -\Big(\lambda^2 F(\lambda) - \lambda\Big) = -\lambda^2 F(\lambda) + \lambda \implies F(\lambda) + \lambda^2 F(\lambda) = \lambda \\ &\implies F(\lambda)\cdot\Big(1+\lambda^2\Big) = \lambda \implies F(\lambda) = \boxed{\frac{\lambda}{1+\lambda^2}} \end{aligned}$$

□

Example 1.9 (Similarity rule)**.** Find $\mathcal{L}\left[\cos(at)\right](\lambda)$.

Solution: Applying the similarity rule:

$$\mathcal{L}\left[\cos(at)\right](\lambda) = \frac{1}{a}\frac{\frac{\lambda}{a}}{1+\frac{\lambda^2}{a^2}} = \boxed{\frac{\lambda}{a^2+\lambda^2}}$$

□

Example 1.10 (Shift rule)**.** Find $\mathcal{L}\left[H(t-a)\right](\lambda)$, where $a \geq 0$ is a constant.

Solution: Applying the shift rule:

$$\mathcal{L}\left[H(t-a)\right](\lambda) = \mathcal{L}\left[H(t-a)\cdot 1\right](\lambda) = \mathcal{L}\left[1\right](\lambda)\cdot e^{-a\lambda} = \boxed{\frac{1}{\lambda}e^{-a\lambda}}$$

□

Notice that as a special case of the previous exercise, we have:

$$\mathcal{L}\left[H(t)\right](\lambda) = \frac{1}{\lambda}$$

Example 1.11 (Rule for multiplication by t^n)**.** Find $\mathcal{L}\left[t^2\right](\lambda)$.

Solution:

$$\mathcal{L}\left[t^2\right](\lambda) \;=\; \mathcal{L}\left[t^2 \cdot 1\right](\lambda) \;=\; (-1)^2 \cdot \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left[\mathcal{L}\left[1\right](\lambda)\right] \;=\; \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left[\frac{1}{\lambda}\right] \;=\; \boxed{\frac{2}{\lambda^3}}$$

□

Example 1.12 (Convolution rule)**.** Let $f(t) = H(t-a)$ and $g(t) = t^2$. Find $\mathcal{L}\left[f * g\right](\lambda)$, i.e. the Laplace transform of the convolution of f and g.

Solution: Applying the convolution rule:

$$\begin{aligned}\mathcal{L}\left[f * g\right](\lambda) &= \mathcal{L}\left[f(t)\right](\lambda) \cdot \mathcal{L}\left[g(t)\right](\lambda) \\ &= \mathcal{L}\left[H(t-a)\right](\lambda) \cdot \mathcal{L}\left[t^2\right](\lambda) = \frac{1}{\lambda}e^{-a\lambda} \cdot \frac{2}{\lambda^3} = \boxed{\frac{2}{\lambda^4} \cdot e^{-a\lambda}}\end{aligned}$$

□

Example 1.13 (Attenuation rule)**.** Find $\mathcal{L}\left[e^{ct}\right](\lambda)$, where $c \geq 0$ is a constant.

Solution:

$$\begin{aligned}\mathcal{L}\left[e^{ct}\right](\lambda) &= \mathcal{L}\left[e^{-(-c)t}\right](\lambda) = \mathcal{L}\left[1 \cdot e^{-(-c)t}\right](\lambda) \\ &= \mathcal{L}\left[1\right]\big(\lambda + (-c)\big) = \frac{1}{\lambda + (-c)} = \boxed{\frac{1}{\lambda - c}}\end{aligned}$$

□

Example 1.14 (Linearity)**.** Find $\mathcal{L}\left[\cosh(2t)\right](\lambda)$. Hint: recall that $\cosh(\theta) = \frac{e^{\theta}+e^{-\theta}}{2}$.

Solution:

$$\begin{aligned}\mathcal{L}\left[\cosh(2t)\right](\lambda) &= \mathcal{L}\left[\frac{e^{2t}+e^{-2t}}{2}\right](\lambda) \\ &= \frac{1}{2}\left(\mathcal{L}\left[e^{2t}\right](\lambda) + \mathcal{L}\left[e^{-2t}\right](\lambda)\right) = \frac{1}{2}\left(\frac{1}{\lambda-2} + \frac{1}{\lambda-(-2)}\right) = \boxed{\frac{\lambda}{\lambda^2-4}}\end{aligned}$$

□

Example 1.15 (Rule for multiplication by t^n)**.** Find $\mathcal{L}\left[t\cos(2t)\right](\lambda)$.

Solution:

$$\begin{aligned}\mathcal{L}\left[t\cos(2t)\right](\lambda) = \mathcal{L}\left[t^1\cos(2t)\right](\lambda) &= -1\cdot\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\mathcal{L}\left[\cos(2t)\right](\lambda)\right]\\ &= -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\lambda}{4+\lambda^2}\right] = -\frac{\lambda^2+4-\lambda\cdot 2\lambda}{(\lambda^2+4)^2} = \boxed{\frac{\lambda^2-4}{(\lambda^2+4)^2}}\end{aligned}$$

□

Chapter 2

The inverse Laplace transform

In section 2.1, we introduce the inverse Laplace transform. In section 2.2, we discuss the concepts of poles and residues, which we will need for the remainder of the chapter. In section 2.3 and section 2.4, we discuss the residue method, which is a way of finding the inverse Laplace transform of a function. In section 2.5 and section 2.6, we discuss a shortcut that can be used to save time when finding residues that involve complex conjugates.

2.1 Definition of the inverse Laplace transform

In this section, we introduce the inverse Laplace transform.

The inverse Laplace transform of a function $F(\lambda)$ is the function $f(t)$ that has $F(\lambda)$ as its Laplace transform. More formally:

Definition 2.1 (Inverse Laplace transform of a function)**.** Let $F(\lambda)$ be a function. Then the *inverse Laplace transform* of $F(\lambda)$, denoted $\mathcal{L}^{-1}[F(\lambda)](t)$, is the function $f(t)$ that satisfies:

$$\mathcal{L}\left[f(t)\right](\lambda) = F(\lambda)$$

Sometimes in math, operations are "multivalued." For example, 9 has two square roots: $+3$ and -3. So you may be asking, "Can a function $F(\lambda)$ have two different inverse Laplace transforms?" Luckily for us, the answer is no.

Theorem 2.2 (Uniqueness of inverse Laplace transform)**.** *Let $f_1(t)$ and $f_2(t)$ be defined for $t \geq 0$. If*

$$\mathcal{L}\left[f_1(t)\right](\lambda) = \mathcal{L}\left[f_2(t)\right](\lambda) = F(\lambda)$$

then $f_1(t)$ and $f_2(t)$ are the same[1] *function.*

[1]So technically, they might differ at a set of isolated points. But this is a minor point that you won't need to worry about in this class.

So we don't need to worry about having to find more than one inverse Laplace transform of the same function.

2.2 Brief digression: Poles and residues

In this section, we will discuss the concepts of poles and residues, which we'll need in the next section.

2.2.1 Poles

We begin with the definition of a pole.

Definition 2.3 (Poles of a function)**.** Suppose $G(\lambda)$ is a function that can be written as a fraction. Then the *poles* of $G(\lambda)$ are the values of λ for which the denominator of $G(\lambda)$ is equal to zero.

Example 2.4 (Identifying poles)**.** Let $G(\lambda) = \frac{1}{\lambda^3 - 3\lambda^2 + 4}$. What are the poles of $G(\lambda)$?

Solution: Since $G(\lambda)$ can be written as a fraction, we are able to proceed. We can factor the denominator:

$$G(\lambda) = \frac{1}{\lambda^3 - 3\lambda^2 + 4} = \frac{1}{(\lambda - 2)^2 (\lambda + 1)}$$

It is clear that the denominator is zero when $\boxed{\lambda = 2}$ or $\boxed{\lambda = -1}$. So these are the poles of $G(\lambda)$. □

2.2.2 Order of a pole

Not all poles are created equal. This leads us to the concept of the order of a pole.

Definition 2.5 (Order of a pole of a function)**.** Suppose that $G(\lambda)$ has a pole at $\lambda = \lambda_0$. When the denominator of $G(\lambda)$ is factored, then the *order of the pole* at λ_0 is the exponent n of the term $(\lambda - \lambda_0)^n$.

Example 2.6 (Identifying orders of poles)**.** Continuing the previous example, let $G(\lambda) = \frac{1}{\lambda^3 - 3\lambda^2 + 4}$. What is the order of each of the poles of $G(\lambda)$?

Solution: As in the previous example, we factor the denominator:

$$G(\lambda) = \frac{1}{\lambda^3 - 3\lambda^2 + 4} = \frac{1}{(\lambda - 2)^2 (\lambda + 1)} = \frac{1}{(\lambda - 2)^2 (\lambda + 1)^1}$$

Now, for each of the poles, we look at the exponent. So the pole at $\lambda = 2$ has $\boxed{\text{order } 2}$, and the pole at $\lambda = -1$ has $\boxed{\text{order } 1}$. □

2.2.3 Residues

Each pole has a real or complex number associated with it called its "residue".

Definition 2.7 (Residue of a pole of a function)**.** Suppose that $G(\lambda)$ has a pole at $\lambda = \lambda_0$ with order n. Then the *residue* of the pole at λ_0, denoted by $\text{Res}(G; \lambda_0)$, is defined by the following formulas:

- If the pole has order $n = 1$:

$$\text{Res}(G; \lambda_0) := \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0) G(\lambda) \tag{2.1}$$

- If the pole has order $n \geq 2$:

$$\text{Res}(G; \lambda_0) := \frac{1}{(n-1)!} \lim_{\lambda \to \lambda_0} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\lambda^{n-1}} \left[(\lambda - \lambda_0)^n G(\lambda) \right] \tag{2.2}$$

Example 2.8 (Finding residues)**.** Continuing the previous two examples, let $G(\lambda) = \frac{1}{\lambda^3 - 3\lambda^2 + 4}$. What is the residue of each of the poles?

Solution: As we discussed in the previous examples, $G(\lambda)$ has two poles: a pole at $\lambda = -1$ of order 1, and a pole at $\lambda = 2$ of order 2 . We find each of the residues individually.

$\underline{\lambda = -1}$: Since this pole has order 1, we use equation (2.1):

$$\begin{aligned} \text{Res}(G; -1) &= \lim_{\lambda \to -1} (\lambda - (-1)) G(\lambda) = \lim_{\lambda \to -1} (\lambda + 1) \frac{1}{\lambda^3 - 3\lambda^2 + 4} \\ &= \lim_{\lambda \to -1} \cancel{(\lambda + 1)} \frac{1}{(\lambda - 2)^2 \cancel{(\lambda + 1)}} = \lim_{\lambda \to -1} \frac{1}{(\lambda - 2)^2} \\ &= \frac{1}{\left((-1) - 2\right)^2} = \frac{1}{(-3)^2} = \frac{1}{9} \end{aligned}$$

So the residue of the pole at $\lambda = -1$ is $\boxed{\text{Res}(G; -1) = \frac{1}{9}}$.

$\underline{\lambda = 2}$: Since this pole has order 2, we have to use equation (2.2):

$$\begin{aligned} \text{Res}(G; 2) &= \frac{1}{(2-1)!} \lim_{\lambda \to 2} \frac{\mathrm{d}^{2-1}}{\mathrm{d}\lambda^{2-1}} \left[(\lambda - 2)^2 G(\lambda) \right] = \lim_{\lambda \to 2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[(\lambda - 2)^2 G(\lambda) \right] \\ &= \lim_{\lambda \to 2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[(\lambda - 2)^2 \frac{1}{\lambda^3 - 3\lambda^2 + 4} \right] = \lim_{\lambda \to 2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[\cancel{(\lambda - 2)^2} \frac{1}{\cancel{(\lambda - 2)^2} (\lambda + 1)} \right] \\ &= \lim_{\lambda \to 2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[\frac{1}{\lambda + 1} \right] = \lim_{\lambda \to 2} -\frac{1}{(\lambda + 1)^2} = -\frac{1}{(2+1)^2} = -\frac{1}{9} \end{aligned}$$

So the residue of the pole at $\lambda = 2$ is $\boxed{\text{Res}(G; 2) = -\frac{1}{9}}$. □

It is often convenient to make a table listing the poles, their orders, and their residues:

Poles of $F(\lambda)$	*Orders of poles*	*Residues of* $F(\lambda)$ *at poles*
-1	1	$\frac{1}{9}$
2	2	$-\frac{1}{9}$

2.3 Residue method for inverse Laplace transforms

In the previous section, we introduced the concepts of poles and residues. In this section, we show how the theory of residues can be used to find the inverse Laplace transform of a function.

The general idea is this: we find the poles of our function $F(\lambda)$, calculate the residues of the related function $F(\lambda)e^{\lambda t}$ at each of those poles, and then sum all of those residues up. More formally:

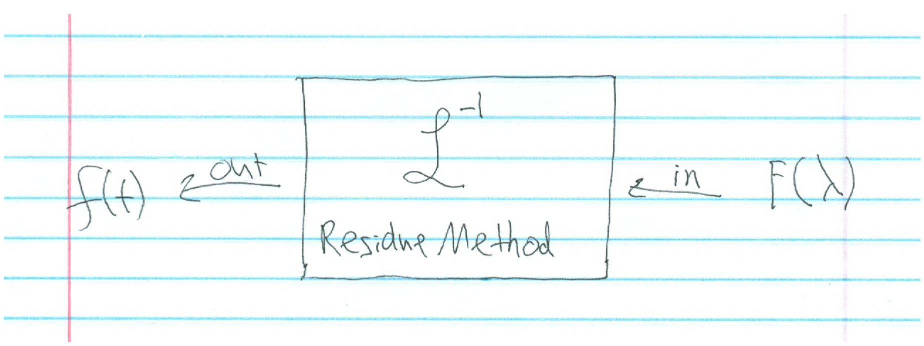
Theorem 2.9 (Residue method)**.** *Suppose that* $F(\lambda)$ *has poles at* $\lambda_1, \cdots, \lambda_k$ *of orders* $n_1, \cdots, n_k$*. Then the inverse Laplace transform of* $F(\lambda)$ *is given by:*

$$f(t) = \mathcal{L}^{-1}[F(\lambda)](t) = \sum_{i=1}^{k} \text{Res}(F(\lambda)e^{\lambda t}; \lambda_i)$$

Warning! It is important to note that although the sum is over all poles of $F(\lambda)$, we don't actually calculate the residues for the function $F(\lambda)$. Instead, we calculate the residues of the slightly different function $F(\lambda)e^{\lambda t}$.

You can think of the residue method as a "machine" that undoes the Laplace transform.

Figure 2.1: The residue method as a metaphorical "machine."



In the next section, we'll do some examples of inverting the Laplace transform via the residue method.

2.4 Examples of residue method

Example 2.10 (Inverse Laplace via residue method)**.** Use the residue method to find the inverse Laplace transform of $F(\lambda) = \frac{1}{\lambda+5}$.

Solution: The denominator is already factored as much as it can be factored. So we see that $F(\lambda)$ has a pole at -5 of order 1. We make a table.

Poles of $F(\lambda)$	*Orders of poles*	*Residues of* $F(\lambda)e^{\lambda t}$ *at poles*
-5	1	

We find the residue of our pole with respect to the function $F(\lambda)e^{\lambda t}$:

$$\begin{aligned}\operatorname{Res}(F(\lambda)e^{\lambda t}; -5) &= \lim_{\lambda\to -5}(\lambda - (-5))F(\lambda)e^{\lambda t} \\ &= \lim_{\lambda\to -5} \cancel{(\lambda+5)}\frac{1}{\cancel{\lambda+5}}e^{\lambda t} = \lim_{\lambda\to -5} e^{\lambda t} = e^{-5t}\end{aligned}$$

We update our table.

Poles of $F(\lambda)$	*Orders of poles*	*Residues of* $F(\lambda)e^{\lambda t}$ *at poles*
-5	1	e^{-5t}

Now, we apply Theorem 2.9.

$$f(t) = \mathcal{L}^{-1}[F(\lambda)](t) = \sum_{i=1}^{k} \operatorname{Res}(F(\lambda)e^{\lambda t}; \lambda_i) = \operatorname{Res}(F(\lambda)e^{\lambda t}; -5) = e^{-5t}$$

So $f(t) = \boxed{e^{-5t}}$. □

Example 2.11 (Inverse Laplace via residue method)**.** Use the residue method to find the inverse Laplace transform of $F(\lambda) = \frac{1}{\lambda^2-3\lambda+2}$.

Solution: We begin by factoring the denominator:

$$F(\lambda) = \frac{1}{\lambda^2 - 3\lambda + 2} = \frac{1}{(\lambda-2)(\lambda-1)}$$

So we see that $F(\lambda)$ has a pole at 2 of order 1 and a pole at 1 of order 1. We make a table.

Poles of $F(\lambda)$	*Orders of poles*	*Residues of* $F(\lambda)e^{\lambda t}$ *at poles*
2	1	
1	1	

We find each of the residues with respect to the function $F(\lambda)e^{\lambda t}$.

$$\begin{aligned}\operatorname{Res}(F(\lambda)e^{\lambda t}; 2) = \lim_{\lambda\to 2}(\lambda-2)F(\lambda)e^{\lambda t} &= \lim_{\lambda\to 2}\cancel{(\lambda-2)}\frac{1}{\cancel{(\lambda-2)}(\lambda-1)}e^{\lambda t} \\ &= \lim_{\lambda\to 2}\frac{1}{\lambda-1}e^{\lambda t} = \frac{1}{2-1}e^{2t} = e^{2t}\end{aligned}$$

Update the table:

Poles of $F(\lambda)$	*Orders of poles*	*Residues of* $F(\lambda)e^{\lambda t}$ *at poles*
2	1	e^{2t}
1	1	

$$\begin{aligned}\text{Res}(F(\lambda)e^{\lambda t};1) &= \lim_{\lambda\to 1}(\lambda-1)F(\lambda)e^{\lambda t} = \lim_{\lambda\to 1}(\lambda-1)\frac{1}{(\lambda-2)(\lambda-1)}e^{\lambda t} \\ &= \lim_{\lambda\to 1}\cancel{(\lambda-1)}\frac{1}{(\lambda-2)\cancel{(\lambda-1)}}e^{\lambda t} = \lim_{\lambda\to 1}\frac{1}{\lambda-2}e^{\lambda t} = \frac{1}{1-2}e^{1t} = -e^{t}\end{aligned}$$

Update the table:

Poles of $F(\lambda)$	*Orders of poles*	*Residues of* $F(\lambda)e^{\lambda t}$ *at poles*
2	1	e^{2t}
1	1	$-e^{t}$

Now we apply Theorem 2.9.

$$\begin{aligned}f(t) = \ &\mathcal{L}^{-1}[F(\lambda)](t) = \sum_{i=1}^{k}\text{Res}(F(\lambda)e^{\lambda t};\lambda_i) \\ &= \text{Res}(F(\lambda)e^{\lambda t};2) + \text{Res}(F(\lambda)e^{\lambda t};1) = e^{2t} - e^{t} = e^{t}\left(e^{t}-1\right)\end{aligned}$$

So $f(t) = \boxed{e^{t}\left(e^{t}-1\right)}$.

□

Example 2.12 (Inverse Laplace via residue method)**.** Use the residue method to find the inverse Laplace transform of $F(\lambda) = \frac{1}{\lambda^2+4}$.

Solution: We begin by factoring the denominator:

$$F(\lambda) = \frac{1}{\lambda^2+4} = \frac{1}{(\lambda+2i)(\lambda-2i)}$$

So we see that $F(\lambda)$ has two poles: a pole at $-2i$ of order 1, and a pole at $2i$ of order 1. We make a table.

Poles of $F(\lambda)$	*Orders of poles*	*Residues of* $F(\lambda)e^{\lambda t}$ *at poles*
$-2i$	1	
$2i$	1	

We find each of the residues with respect to the function $F(\lambda)e^{\lambda t}$.

$$\begin{aligned}\text{Res}(F(\lambda)e^{\lambda t};-2i) &= \lim_{\lambda\to -2i}(\lambda-(-2i))F(\lambda)e^{\lambda t} \\ = \lim_{\lambda\to -2i}\cancel{(\lambda+2i)}\frac{1}{\cancel{(\lambda+2i)}(\lambda-2i)}e^{\lambda t} &= \lim_{\lambda\to -2i}\frac{1}{(\lambda-2i)}e^{\lambda t} = \frac{1}{(-2i-2i)}e^{-2it} \\ &= \frac{1}{-4i}e^{-2it} = \frac{1}{-4i}\frac{i}{i}e^{-2it} = \frac{i}{-4i^2}e^{-2it} = \frac{i}{4}e^{-2it}\end{aligned}$$

Updating our table:

Poles of $F(\lambda)$	*Orders of poles*	*Residues of* $F(\lambda)e^{\lambda t}$ *at poles*
$-2i$	1	$\frac{i}{4}e^{-2it}$
$2i$	1	

$$\text{Res}(F(\lambda)e^{\lambda t}; 2i) = \lim_{\lambda \to 2i} (\lambda - 2i)F(\lambda)e^{\lambda t}$$

$$= \lim_{\lambda \to 2i} \cancel{(\lambda - 2i)} \frac{1}{(\lambda + 2i)\cancel{(\lambda - 2i)}} e^{\lambda t} = \lim_{\lambda \to 2i} \frac{1}{(\lambda + 2i)} e^{\lambda t} = \frac{1}{(2i + 2i)} e^{2it}$$

$$= \frac{1}{4i}e^{2it} = \frac{1}{4i}\frac{i}{i}e^{2it} = \frac{i}{4i^2}e^{2it} = -\frac{i}{4}e^{2it}$$

Updating our table:

Poles of $F(\lambda)$	*Orders of poles*	*Residues of* $F(\lambda)e^{\lambda t}$ *at poles*
$-2i$	1	$\frac{i}{4}e^{-2it}$
$2i$	1	$-\frac{i}{4}e^{2it}$

Now, we apply Theorem 2.9. (It will be helpful to consult Appendix B, which reviews several useful facts about trig functions.)

$$f(t) = \mathcal{L}^{-1}[F(\lambda)](t) = \sum_{i=1}^{k} \text{Res}(F(\lambda)e^{\lambda t}; \lambda_i)$$

$$= \text{Res}(F(\lambda)e^{\lambda t}; -2i) + \text{Res}(F(\lambda)e^{\lambda t}; 2i) = \frac{i}{4}e^{-2it} + -\frac{i}{4}e^{2it}$$

$$= \frac{i}{4}\big(\cos(-2t) + i\sin(-2t)\big) + -\frac{i}{4}\big(\cos(2t) + i\sin(2t)\big)$$

$$= \frac{i}{4}\cos(-2t) + \frac{i^2}{4}\sin(-2t) - \frac{i}{4}\cos(2t) - \frac{i^2}{4}\sin(2t)$$

$$= \frac{i}{4}\cos(2t) - \frac{i^2}{4}\sin(2t) - \frac{i}{4}\cos(2t) - \frac{i^2}{4}\sin(2t)$$

$$= \cancel{\frac{i}{4}\cos(2t)} + \frac{1}{4}\sin(2t) - \cancel{\frac{i}{4}\cos(2t)} + \frac{1}{4}\sin(2t) = \frac{1}{2}\sin 2t$$

So $f(t) = \boxed{\frac{1}{2}\sin 2t}$.

In the next section, we'll discuss a shorter way of doing this particular kind of problem (the kind that involves complex conjugates). □

2.5 Residues of complex conjugates

In this section, we describe a shortcut that can be used to save time when finding residues involving complex conjugates.

Let us first quickly recall two definitions relating to complex numbers.

Definition 2.13 (Complex conjugate)**.** Let $z = x + iy$ be a complex number, where x and y are real numbers. Then the *complex conjugate* of z, denoted $\overline{z}$, is:

$$\overline{z} = \overline{x + iy} = x - iy$$

Definition 2.14 (Real part)**.** Let $z = x + iy$ be a complex number, where x and y are real numbers. Then the *real part* of z, denoted $\Re\{z\}$ is:

$$\Re\{z\} = \Re\{x + iy\} = x$$

If we have two poles that are complex conjugates, then we can save time. Instead of finding the residues at each of the poles separately, we can find the residue of one of the poles, take its real part, and double it. More formally:

Theorem 2.15 (Shortcut for residues of complex conjugates)**.** *Suppose $F(\lambda)$ has a pole of order 1 at $\lambda_1 = x + iy$ and a pole of order 1 at $\lambda_2 = x - iy$. Then:*

$$\text{Res}(F(\lambda)e^{\lambda t}; \lambda_1) + \text{Res}(F(\lambda)e^{\lambda t}; \lambda_2) = 2\,\Re\left\{\text{Res}(F(\lambda)e^{\lambda t}; \lambda_1)\right\}$$

In the next section, we'll look at some examples of using this shortcut.

2.6 Examples of complex conjugate shortcut

Example 2.16 (Residues of complex conjugates)**.** Use the residue method to find the inverse Laplace transform of $F(\lambda) = \frac{1}{\lambda^2+4}$.

Solution: This is the same problem we did in Example 2.12. However, this time we'll use the shortcut from Theorem 2.15 to get the same answer with less work. We begin by factoring the denominator:

$$F(\lambda) = \frac{1}{\lambda^2 + 4} = \frac{1}{(\lambda - 2i)(\lambda + 2i)}$$

So we see that $F(\lambda)$ has a pole at $2i$ of order 1 and a pole at $-2i$ of order 1. We make a table.

Poles of $F(\lambda)$	*Orders of poles*	*Residues of $F(\lambda)e^{\lambda t}$ at poles*
$2i$, $-2i$	1 each	

Now let's find the residue of $F(\lambda)e^{\lambda t}$ at $2i$: (it doesn't matter which pole we pick)

$$\text{Res}(F(\lambda)e^{\lambda t}; 2i) = \lim_{\lambda \to 2i} (\lambda - 2i) F(\lambda) e^{\lambda t}$$

$$= \lim_{\lambda \to 2i} \cancel{(\lambda - 2i)} \frac{1}{(\lambda + 2i)\cancel{(\lambda - 2i)}} e^{\lambda t} = \lim_{\lambda \to 2i} \frac{1}{(\lambda + 2i)} e^{\lambda t} = \frac{1}{(2i + 2i)} e^{2it}$$

$$= \frac{1}{4i} e^{2it} = \frac{1}{4i} \frac{i}{i} e^{2it} = \frac{i}{4i^2} e^{2it} = -\frac{i}{4} e^{2it}$$

Now we take twice the real part, giving us the sum of the residues at $2i$ and $-2i$:

$$\begin{aligned}
\operatorname{Res}(F(\lambda)e^{\lambda t};2i)+\operatorname{Res}(F(\lambda)e^{\lambda t};-2i) &= 2\,\Re\left\{\operatorname{Res}(F(\lambda)e^{\lambda t};2i)\right\} \\
&= 2\,\Re\left\{-\frac{i}{4}e^{2it}\right\} = 2\,\Re\left\{-\frac{i}{4}\left(\cos(2t)+i\sin(2t)\right)\right\} \\
&= 2\,\Re\left\{-\frac{i}{4}\cos(2t)-\frac{i^2}{4}\sin(2t)\right\} = 2\,\Re\left\{-\frac{i}{4}\cos(2t)+\frac{1}{4}\sin(2t)\right\} \\
&= 2\cdot\frac{1}{4}\sin(2t) = \frac{1}{2}\sin(2t)
\end{aligned}$$

We update our table:

Poles of $F(\lambda)$	*Orders of poles*	*Residues of $F(\lambda)e^{\lambda t}$ at poles*
$2i$, $-2i$	1 each	$\frac{1}{2}\sin(2t)$

Applying Theorem 2.9 gives us our final answer:

$$f(t) = \mathcal{L}^{-1}[F(\lambda)](t) = \sum_{i=1}^{k}\operatorname{Res}(F(\lambda)e^{\lambda t};\lambda_i)$$

$$= \operatorname{Res}(F(\lambda)e^{\lambda t};-2i)+\operatorname{Res}(F(\lambda)e^{\lambda t};2i) = 2\,\Re\left\{\operatorname{Res}(F(\lambda)e^{\lambda t};2i)\right\}` = \frac{1}{2}\sin(2t)$$

So $f(t) = \boxed{\frac{1}{2}\sin 2t}$. This is the same answer we found in Example 2.12, but we did less work. □

Example 2.17 (Residues of complex conjugates)**.** Let $F(\lambda) = \frac{2\lambda+1}{\lambda^2+1}$. Find $f(t)$, the inverse Laplace transform of $F(\lambda)$.

Solution: We begin by factoring the denominator:

$$F(\lambda) = \frac{2\lambda+1}{\lambda^2+1} = \frac{2\lambda+1}{(\lambda+i)(\lambda-i)}$$

So we see that $F(\lambda)$ has two poles: a pole at $\lambda = i$ with order 1, and a pole at $\lambda = -i$ of order 1.

Poles of $F(\lambda)$	*Orders of poles*	*Residues of $F(\lambda)e^{\lambda t}$ at poles*
i, $-i$	1 each	

Now let's find the residues of $F(\lambda)e^{\lambda t}$ at $-i$:

$$\begin{aligned}
\operatorname{Res}(F(\lambda)e^{\lambda t};-i) &= \lim_{\lambda\to-i}(\lambda+i)F(\lambda)e^{\lambda t} \\
&= \lim_{\lambda\to-i}\cancel{(\lambda+i)}\frac{2\lambda+1}{\cancel{(\lambda+i)}(\lambda-i)}e^{\lambda t} = \lim_{\lambda\to-i}\frac{2\lambda+1}{\lambda-i}e^{\lambda t} \\
&= \left(\frac{2(-i)+1}{-i-i}\right)e^{-it} = \left(\frac{-2i+1}{-2i}\right)e^{-it} = \left(1-\frac{1}{2i}\right)e^{-it} = \left(1+\frac{i}{2}\right)e^{-it}
\end{aligned}$$

Now we take twice the real part, giving us the sum of the residues at i and $-i$:

$$\begin{aligned}
&\operatorname{Res}(F(\lambda)e^{\lambda t}; i) + \operatorname{Res}(F(\lambda)e^{\lambda t}; -i) = 2\,\Re\left\{\operatorname{Res}(F(\lambda)e^{\lambda t}; -i)\right\} \\
&\quad = 2\,\Re\left\{\left(1+\frac{i}{2}\right)e^{-it}\right\} = 2\,\Re\left\{\left(1+\frac{i}{2}\right)\left(\cos(-t)+i\sin(-t)\right)\right\} \\
&\quad = 2\,\Re\left\{\cos(-t)+i\sin(-t)+\frac{i}{2}\cos(-t)+\frac{i^2}{2}\sin(-t)\right\} \\
&\quad = 2\,\Re\left\{\cos(t)-i\sin(t)+\frac{i}{2}\cos(t)+\frac{1}{2}\sin(t)\right\} \\
&\quad = 2\cdot\left(\cos(t)+\frac{1}{2}\sin(t)\right) = 2\cos(t)+\sin(t)
\end{aligned}$$

We update our table.

Poles of $F(\lambda)$	*Orders of poles*	*Residues of* $F(\lambda)e^{\lambda t}$ *at poles*
$i,\ -i$	1 each	$2\cos(t)+\sin(t)$

Applying Theorem 2.9 gives us our final answer:

$$f(t) = \mathcal{L}^{-1}[F(\lambda)](t) = \sum_{i=1}^{k} \operatorname{Res}(F(\lambda)e^{\lambda t}; \lambda_i)$$

$$= \operatorname{Res}(F(\lambda)e^{\lambda t}; -i) + \operatorname{Res}(F(\lambda)e^{\lambda t}; i) = 2\,\Re\left\{\operatorname{Res}(F(\lambda)e^{\lambda t}; -i)\right\} = 2\cos(t)+\sin(t)$$

So $f(t) = \boxed{2\cos(t)+\sin(t)}$. □

Chapter 3

Initial value problems

In this chapter, we finally connect Laplace transforms to differential equations. In section 3.1, we explain how to use the Laplace transform to solve initial value problems. In section 3.2, we do several examples.

3.1 Using Laplace transforms to solve initial value problems

We can use the Laplace transform to solve an initial value problem. The general set of steps is this:

Procedure 3.1 (Solving IVPs with Laplace transform)**.** Suppose we have an initial value problem of the form:

$$\begin{cases} f_n(t)y^{(n)}(t) + \cdots + f_1(t)y'(t) + f_0(t)y(t) = g(t) \\ y(0) = a_0 \\ \vdots \\ y^{(n)}(0) = a_n \end{cases}$$

(The first line is our ordinary differential equation, and the subsequent lines are the initial conditions.) We can solve this initial value problem by performing the following steps:

Step 1: Take the Laplace transform of both sides of the ordinary differential equation.

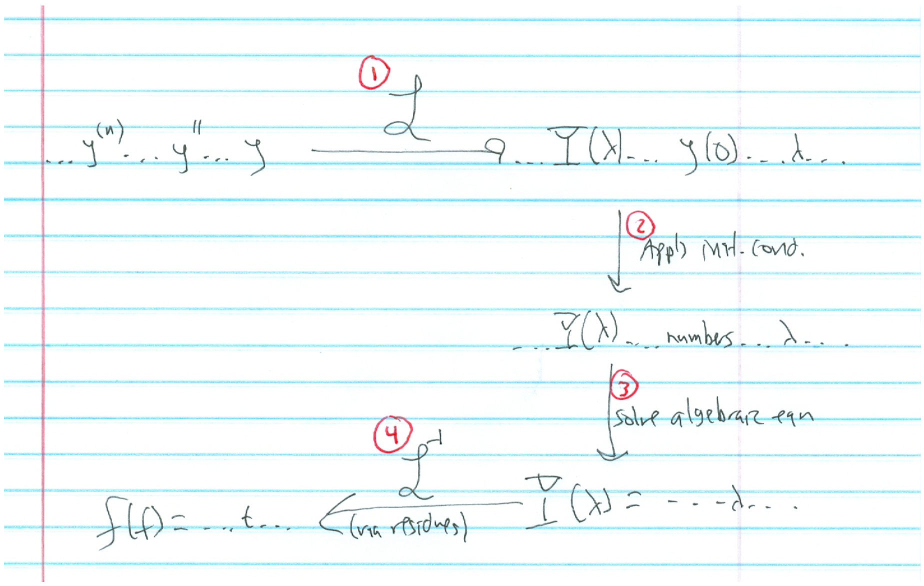
Step 2: Substitute in the initial conditions.

Step 3: Solve for $Y(\lambda)$ in terms of λ

Step 4: Use the residue method to find $y(t) = \mathcal{L}^{-1}\left[Y(\lambda)\right]$

The following diagram illustrates Procedure 3.1.

Figure 3.1: Solving an IVP with Laplace transforms.



3.2 Examples

In this section we do some examples of how to apply Procedure 3.1 to solve an IVP.

Example 3.2 (Using Laplace to solve IVP)**.** Use Laplace transforms to solve the following initial value problem:

$$\begin{cases} y''(t) + y(t) = 1 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Solution: Step 1: We take the Laplace transform of both sides of our ODE:

$$\begin{aligned} y''(t) + y(t) = 1 &\implies \mathcal{L}\left[y''(t) + y(t)\right] = \mathcal{L}\left[1\right] \\ &\implies \mathcal{L}\left[y''(t)\right] + \mathcal{L}\left[y(t)\right] = \mathcal{L}\left[1\right] \\ &\implies \lambda^2 Y(\lambda) - \lambda y(0) - y'(0) + Y(\lambda) = \frac{1}{\lambda} \end{aligned}$$

Step 2: We substitute in the initial conditions for $y(0)$ and $y'(0)$:

$$\lambda^2 Y(\lambda) - \lambda y(0) - y'(0) + Y(\lambda) = \frac{1}{\lambda}$$
$$\implies \lambda^2 Y(\lambda) - \lambda \cdot 0 - 0 + Y(\lambda) = \frac{1}{\lambda} \implies \lambda^2 Y(\lambda) + Y(\lambda) = \frac{1}{\lambda}$$

Step 3: We now solve our algebraic equation for $Y(\lambda)$:

$$\lambda^2 Y(\lambda) + Y(\lambda) = \frac{1}{\lambda} \implies Y(\lambda) \cdot \left(\lambda^2 + 1\right) = \frac{1}{\lambda} \implies Y(\lambda) = \frac{1}{\lambda} \cdot \frac{1}{\lambda^2 + 1}$$
$$\implies Y(\lambda) = \frac{1}{\lambda\left(\lambda^2+1\right)}$$

Step 4: Now, we need to use the residue method to invert the Laplace transform. We begin by factoring the denominator of $Y(\lambda)$:

$$Y(\lambda) = \frac{1}{\lambda\left(\lambda^2+1\right)} = \frac{1}{\lambda(\lambda+i)(\lambda-i)}$$

So $Y(\lambda)$ has three poles: a pole of order 1 at 0, a pole of order 1 at $-i$, and a pole of order 1 at i. We make a table.

Poles of $F(\lambda)$	*Orders of poles*	*Residues of $F(\lambda)e^{\lambda t}$ at poles*
0	1	
$i, -i$	1 each	

First, we find the residue of $Y(\lambda)e^{\lambda t}$ at 0:

$$\operatorname{Res}(Y(\lambda)e^{\lambda t}; 0) = \lim_{\lambda\to 0}\left((\lambda - 0)Y(\lambda)e^{\lambda t}\right) = \lim_{\lambda\to 0}\left(\cancel{\lambda} \cdot \frac{1}{\cancel{\lambda}(\lambda+i)(\lambda-i)}e^{\lambda t}\right)$$
$$= \lim_{\lambda\to 0}\left(\frac{1}{(\lambda+i)(\lambda-i)}e^{\lambda t}\right) = \left(\frac{1}{(0+i)(0-i)}e^{0t}\right) = \left(\frac{1}{(i)(-i)}\right) = 1$$

Updating our table:

Poles of $F(\lambda)$	*Orders of poles*	*Residues of $F(\lambda)e^{\lambda t}$ at poles*
0	1	1
$i, -i$	1 each	

We now need to find the sum of the residues at i and $-i$. Using the shortcut described in Theorem 2.15, we only need to find one of the residues, and then take twice its real part. First we find the residue at i:

$$\operatorname{Res}(Y(\lambda)e^{\lambda t}; i) = \lim_{\lambda\to i}\left((\lambda - i)Y(\lambda)e^{\lambda t}\right) = \lim_{\lambda\to i}\left(\cancel{(\lambda-i)}\frac{1}{\lambda(\lambda+i)\cancel{(\lambda-i)}}e^{\lambda t}\right)$$
$$= \lim_{\lambda\to i}\left(\frac{1}{\lambda(\lambda+i)}e^{\lambda t}\right) = \left(\frac{1}{i(i+i)}e^{it}\right) = \left(\frac{1}{i(2i)}e^{it}\right) = -\frac{1}{2}e^{it}$$

So $\text{Res}(Y(\lambda)e^{\lambda t}; i) = -\frac{1}{2}e^{it}$. Now we take twice its real part:

$$\begin{aligned}
&\text{Res}(Y(\lambda)e^{\lambda t}; i) + \text{Res}(F(\lambda)e^{\lambda t}; -i) = 2\,\Re\left\{\text{Res}(F(\lambda)e^{\lambda t}; i)\right\} \\
&= 2\,\Re\left\{-\frac{1}{2}e^{it}\right\} = 2\,\Re\left\{-\frac{1}{2}(\cos(t) + i\sin(t))\right\} = 2 \cdot -\frac{1}{2}(\cos(t) = -\cos(t)
\end{aligned}$$

Updating our table:

Poles of $F(\lambda)$	*Orders of poles*	*Residues of* $F(\lambda)e^{\lambda t}$ *at poles*
0	1	1
i, $-i$	1 each	$-\cos(t)$

Now we apply Theorem 2.9:

$$\begin{aligned}
y(t) = \mathcal{L}^{-1}[Y(\lambda)](t) &= \sum_{i=1}^{k} \text{Res}(Y(\lambda)e^{\lambda t}; \lambda_i) \\
&= \text{Res}(Y(\lambda)e^{\lambda t}; 0) + \text{Res}(Y(\lambda)e^{\lambda t}; i) + \text{Res}(Y(\lambda)e^{\lambda t}; -i) \\
&= \text{Res}(Y(\lambda)e^{\lambda t}; 0) + 2\,\Re\left\{\text{Res}(F(\lambda)e^{\lambda t}; i)\right\} = 1 - \cos(t)
\end{aligned}$$

So the final solution to our initial value problem is $\boxed{y(t) = 1 - \cos(t)}$.

□

Example 3.3 (Using Laplace to solve IVP)**.** Use Laplace transforms to solve the following initial value problem:

$$\begin{cases} y''(t) - 2y'(t) + 2y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Solution: Step 1: We take the Laplace transform of both sides of our ODE:

$$\begin{aligned}
y''(t) - 2y'(t) + 2y = 0 &\implies \mathcal{L}\left[y''(t) - 2y'(t) + 2y\right] = \mathcal{L}\left[0\right] \\
&\implies \mathcal{L}\left[y''(t)\right] - 2\mathcal{L}\left[y'(t)\right] + 2\mathcal{L}\left[y\right] = \mathcal{L}\left[0\right] \\
&\implies \lambda^2 Y(\lambda) - \lambda y(0) - y'(0) - 2\left(\lambda Y(\lambda) - y(0)\right) + 2Y(\lambda) = 0 \\
&\implies \lambda^2 Y(\lambda) - \lambda y(0) - y'(0) - 2\lambda Y(\lambda) + 2y(0) + 2Y(\lambda) = 0
\end{aligned}$$

Step 2: We substitute in the initial conditions for $y(0)$ and $y'(0)$:

$$\begin{aligned}
&\lambda^2 Y(\lambda) - \lambda y(0) - y'(0) - 2\lambda Y(\lambda) + 2y(0) + 2Y(\lambda) = 0 \\
&\implies \lambda^2 Y(\lambda) - \lambda \cdot 0 - 1 - 2\lambda Y(\lambda) + 2 \cdot +2Y(\lambda) = 0 \\
&\implies \lambda^2 Y(\lambda) - 2\lambda Y(\lambda) + 2Y(\lambda) - 1 = 0
\end{aligned}$$

Step 3: We now solve our algebraic equation for $Y(\lambda)$:

$$\lambda^2 Y(\lambda) - 2\lambda Y(\lambda) + 2Y(\lambda) - 1 = 0 \implies \cdots \implies Y(\lambda) = \frac{1}{\lambda^2 - 2\lambda + 2}$$

Step 4: Now, we need to use the residue method to invert the Laplace transform. We begin by factoring[1] the denominator of $Y(\lambda)$:

$$Y(\lambda) = \frac{1}{\lambda^2 - 2\lambda + 2} = \frac{1}{(\lambda - (1+i))(\lambda - (1-i))}$$

So $Y(\lambda)$ has a pole at $1+i$ of order 1 and a pole at $1-i$ of order 1. We make a table.

Poles of $F(\lambda)$	*Orders of poles*	*Residues of $F(\lambda)e^{\lambda t}$ at poles*
$1+i$, $1-i$	1 each	

We now need to find the sum of the residues at $1+i$ and $1-i$. Using the shortcut described in Theorem 2.15, we only need to find one of the residues, and then take twice its real part. We'll find the residue at $1+i$.

I'll let you do the algebra of finding the residue on your own. Here's what you should get:

$$\operatorname{Res}(Y(\lambda)e^{\lambda t}; 1+i) = \cdots = -\frac{1}{2} i e^{(1+i)t}$$

Now we take twice the real part. Again, I'll let you do the algebra. Here's what you should get:

$$2\,\mathfrak{R}\left\{-\frac{1}{2} i e^{(1+i)t}\right\} = \cdots = 2\,\mathfrak{R}\left\{-\frac{1}{2} i e^t \cos(t) + \frac{1}{2} e^t \sin(t)\right\}$$
$$= 2 \cdot \frac{1}{2} e^t \sin(t) = e^t \sin(t)$$

We fill in our table:

Poles of $F(\lambda)$	*Orders of poles*	*Residues of $F(\lambda)e^{\lambda t}$ at poles*
$1+i$, $1-i$	1 each	$e^t \sin(t)$

Now we apply Theorem 2.9:

$$y(t) = \mathcal{L}^{-1}[Y(\lambda)](t) = \sum_{i=1}^{k} \operatorname{Res}(Y(\lambda)e^{\lambda t}; \lambda_i) = e^t \sin(t)$$

So the final solution to our initial value problem is $\boxed{y(t) = e^t \sin(t)}$. □

[1]Sometimes, it's not immediately obvious how the denominator will factor. In those cases, it's often helpful to write out the quadratic formula on a scratch sheet of paper: $\lambda = \frac{-(-2)\pm\sqrt{(-2)^2-4\cdot1\cdot2}}{2\cdot1} = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm2i}{2} = 1 \pm i$

Appendix A

Table of common Laplace transforms

Table A-1: Commonly encountered functions and their Laplace transforms

$f(t)$	$F(\lambda)$
1	$\frac{1}{\lambda}$
$H(t-a)$	$\frac{1}{\lambda}e^{-a\lambda}$
e^{at}	$\frac{1}{\lambda-a}$
t^n	$\frac{n!}{\lambda^{n+1}}$
$t^n e^{at}$	$\frac{n!}{(\lambda-a)^{n+1}}$
$\sin(at)$	$\frac{a}{\lambda^2+a^2}$
$\cos(at)$	$\frac{\lambda}{\lambda^2+a^2}$
$\delta(t-a)$	$e^{-a\lambda}$

Appendix B

Useful facts about trigonometric functions

Formula B.1 (Euler's identity)**.**

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Formula B.2 (Sine is odd function)**.**

$$\sin(-\theta) = -\sin(\theta)$$

Formula B.3 (Cosine is even function)**.**

$$\cos(-\theta) = \cos(\theta)$$

Formula B.4 (Sine angle addition formula)**.**

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\beta)\sin(\alpha)$$

Formula B.5 (Cosine angle addition formula)**.**

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$