# On Adomian's Decomposition Method for Solving Differential Equations 

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#### Abstract

We show that with a few modifications the Adomian's method for solving second order differential equations can be used to obtain the known results of the special functions of mathematical physics. The modifications are necessary in order to take corectly into account the behavior of the solutions in the neighborhood of the singular points.


## 1 Introduction

Recently a great deal of interest has been focused on the application of Adomian's decomposition method to solve a wide variety of stochastic and deterministic problems [1].Although the Adomian's goal is to find a method to unify linear and nonlinear, ordinary or partial differential equations for solving initial and boundary value problems, we shall deal in the following only with linear second order differential equations.

Our aim here is to compare the decomposition method with the classical methods for solving differential equations in order to obtain a better understanding of it. Because the special functions are extremely useful tools for obtaining closed form as well as series solutions to a variety of problems arising in science and engineering we tryed to reobtain the known results by the new method. We found that with a slight modification the method works also in this case.

Usually one starts with an equation $F f(x)=g(x)$, where $F$ represents a general nonlinear ordinary differential operator. $F$ is decomposed in two
parts $L+R$, where $L$ is the linear part of $F$. The core of the method consists in finding an inverse of $L$. In many cases this is not possible, in others the Green function is quite complicate and consequently there are difficulties of integration. The solution proposed by Adomian [1] is to take $L$ as the heighest-ordered derivative of the linear part. For example, for initial value problems Adomian defines $L^{-1}$ for $L=d^{2} / d x^{2}$ as the two-fold integration operator from 0 to $x$. We have $L^{-1} L f(x)=f(x)-f(0)-x f^{\prime}(0)$ and therefore

$$
f(x)=f(0)+x f^{\prime}(0)-L^{-1} R+L^{-1} g
$$

Such stated the method is quite old. For linear second order differential equations it has been proposed by Cochran, who transformed the equation

$$
\left[p(x) f^{\prime}(x)\right]^{\prime}+q(x) f(x)=g(x)
$$

into an equivalent Volterra integral equation [2]. Cochran has been interested in the uniqueness of solutions for two-point boundary-value problems and in the existence of eigenfunctions of the related homogeneous equation.

Adomian's point of view was to take $L$ as the simplest easily invertible operator pushing all the other terms in the reminder $R$ or in the non- homogeneous part. With this choice the method has given many beautiful results.

Our first comment here is that in order to find the classical results for the special functions this approach is not sufficient. The second one is that we give an example, the Bessel polynomials equation, where $L$ has to be chosen as a first order differential operator for solving the equation, although the equation satisfied by them is of second order. The last comment points out that the approach 1.1 supposes that $f(x)$ and its derivative are finite at the point $x=0$, thus one cannot obtain solutions in the neighborhood of a singular point.

As a first conclusion this means that the decomposition method is a useful tool for solving concrete problems, but clearly its form cannot be decided independent of the problem.

We shall consider the linear second order equation written in the more general form

$$
L(x, D) f(x)-R\left(x, f, D f, D^{2} f\right)=0
$$

where $D=d / d x$ and $R$ is the reminder which includes all the other terms not contained in the principal part $L(x, D)$ whose form is

$$
L(x, D)=h(x) D p(x) D
$$

$h(x)$ and $p(x)$ are smooth functions whose properties are given below. The minus sign in 1.2 is taken for convenience.

At this stage the clue of the problem is to make a clever choice of $L$ and $R$ such that the resulting pseudo Volterra integral equation should be easily solvable.

In many cases bringing 1.2 to the canonical form complicates the matter. We propose to find an inverse for the general form 1.3 and use it to find many of the known formulas for the solutions of classical special function equations.

Our second approach consists in writting 1.2 as a formal non-homogeneous equation and by using a variant of the method of variation of parameters we transform it into a pseudo Volterra integral equation. We consider this form as the most general form of the decomposition method for the second order differential equations. We apply it to solve the Bessel functions equation.

## 2 Special Functions Equations

Most of the equations for the classical special functions are of the form 1.2, with the principal part $L(x, D)$ of the form 1.3. Many of them have $h(x)=1$.

The functions $h(x)$ and $p(x)$ are supposed to be smooth and $1 / p(x)$ locally integrable around some point, which we can take to be $x=0$ without loss of generality. The decomposition method consists in finding an inverse operator $L^{-1}$ and with its help we are able to construct in many cases the exact solution of the problem.

A formal inverse of 1.3 can be easily found. We choose it as

$$
L^{-1}(x, D) f(x)=\int_{0}^{x} \frac{d t}{p(t)} \int_{0}^{t} \frac{d y}{h(y)} f(y)
$$

One sees that

$$
\left(L L^{-1}\right) f(x)=I \cdot f(x)
$$

but

$$
\left(L^{-1} L\right) f(x) \neq I \cdot f(x)
$$

where $I$ is the identity operator. The second relation tell us that $L^{-1}$ is not a true inverse, it becomes so when we take into account the initial conditions, which is equivalent with prescribing the values of $f(0)$ and $f^{\prime}(0)$.

Indeed

$$
\begin{gathered}
\left(L^{-1} L\right) f(x)=\int_{0}^{x} \frac{d t}{p(t)} \int_{0}^{t} \frac{d y}{h(y)} h(y) \frac{d}{d y} p(y) \frac{d f(y)}{d y} \\
=\int_{0}^{x} \frac{d t}{p(t)}\left[p(y) \frac{d f}{d y}\right]_{0}^{t}=\int_{0}^{x} \frac{d f(t)}{d t}-p(0) f^{\prime}(0) \int_{0}^{x} \frac{d t}{p(t)} \\
=f(x)-f(0)-p(0) f^{\prime}(0) \int_{0}^{x} \frac{d t}{p(t)}
\end{gathered}
$$

Thus we obtain from 1.2

$$
f(x)=f(0)-p(0) f^{\prime}(0) \int_{0}^{x} \frac{d t}{p(t)}+\int_{0}^{x} \frac{d t}{p(t)} \int_{0}^{t} \frac{d y}{h(y)} R(y, D f)
$$

which is a Volterra integral equation.. Its most general form for second order differential equations will be given in Section 3. The solution of the last equation is sought as a series

$$
f(x)=\sum_{k=0}^{\infty} f_{k}(x)
$$

where

$$
f_{0}=f(0)-p(0) f^{\prime}(0) \int_{0}^{x} \frac{d t}{p(t)}
$$

and $f_{k}(x)$ is obtained by the Picard method of succesive approximation

$$
f_{k}=L^{-1}(x, D) f_{k-1}
$$

In the following we shall apply the above technique to a few classical equations.

### 2.1 Gauss Hypergeometric Function

Because the solutions to many mathematical physics problems are expressed by the hypergeometric functions we consider first this equation whose standard form is

$$
x(1-x) f^{\prime \prime}(x)+[\gamma-(\alpha+\beta+1) x] f^{\prime}(x)-\alpha \beta f(x)=0
$$

and rewrite it as

$$
\begin{aligned}
& L f(x)=x f^{\prime \prime}(x)+\gamma f^{\prime}(x) \equiv x^{1-\gamma} \frac{d}{d x}\left(x^{\gamma} \frac{d f}{d x}\right) \\
& =x^{2} f^{\prime \prime}(x)+(\alpha+\beta+1) x f^{\prime}(x)+\alpha \beta f(x)
\end{aligned}
$$

In this case $h(x)=x^{1-\gamma}$ and $p(x)=x^{\gamma}$. We suppose that $\gamma>0$ and then

$$
\left(L^{-1} f\right)(x)=f(x)-f(0)
$$

We take as usual in the decomposition method

$$
f(x)=f_{o}(x)+f_{1}(x)+f_{2}(x)+\ldots
$$

and choose the initial condition as

$$
f_{0}=f(0)=1
$$

Using the form 2.2 we get the recurrence relation

$$
\begin{gathered}
f_{n}=L^{-1}\left[x^{2} f_{n-1}^{\prime \prime}(x)+(\alpha+\beta+1) x f_{n-1}^{\prime}(x)+\alpha \beta f_{n-1}(x)\right] \\
=\int_{0}^{x} x^{-\gamma} d x \int_{0}^{x} x^{\gamma-1}\left[x^{2} f_{n-1}^{\prime \prime}(x)+(\alpha+\beta+1) x f_{n-1}^{\prime}(x)+\alpha \beta f_{n-1}(x)\right] d x
\end{gathered}
$$

We find that

$$
f_{1}=\frac{\alpha \beta}{\gamma} \frac{x}{1!}
$$

and by induction

$$
f_{n}=\frac{(\alpha)_{n}(\beta)_{n}}{(\gamma)_{n}} \frac{x^{n}}{n!}
$$

where $(\alpha)_{n}=\alpha(\alpha+1) \ldots(\alpha+n-1)$
In this way we get the well known solution

$$
f(x)={ }_{2} F_{1}(\alpha, \beta ; \gamma ; x)=1+\sum_{n=1}^{\infty} \frac{(\alpha)_{n}(\beta)_{n}}{(\gamma)_{n}} \frac{x^{n}}{n!}
$$

### 2.2 Confluent Hypergeometric Functions

As concerns the confluent hypergeometric equation

$$
x f^{\prime \prime}(x)+(\gamma-x) f^{\prime}(x)-\alpha f(x)=0
$$

its solution can be obtained in the same manner as that of the Gauss hypergeometric function, the functions $h(x)$ and $p(x)$ being the same.

We shall consider now the equation

$$
f^{\prime \prime}(x)+a x^{q-1} f^{\prime}(x)+b x^{q-2} f(x)=0
$$

where $a, b$ and $q$ are complex numbers, whose solution seems to be unknown. For $q=0$ it reduces to Euler equation. For others values of $q$ it is equivalent with a degenerate hypergeometric function. More about this equation see (3), eq. 2.60].

We construct two independent solutions in the neighborhood of the point $x=0$. We take $L=D^{2}$ and then

$$
L^{-1} f(x)=f(x)-f(0)-x f^{\prime}(0)
$$

The first solution is defined by

$$
f(0)=1, \quad f^{\prime}(0)=0
$$

We choose

$$
f_{0}=f(0)=1
$$

and find

$$
f_{1}=-\frac{b}{q(q-1)} x^{q}
$$

We make the ansatz

$$
f_{k}=c_{k} x^{k q}
$$

and from the relation $f_{k+1}=L^{-1} f_{k}$ we get

$$
\begin{gathered}
f_{k+1}=-c_{k} \int_{0}^{x} d x \int_{0}^{x}\left[a q k x^{(k+1) q-2}+b x^{(k+1) q-2}\right] \\
=-c_{k} \frac{a q k+b}{(k+1) q-1} \frac{x^{(k+1) q}}{q(k+1)}
\end{gathered}
$$

In this way we find the recurrence relation

$$
c_{k+1}=-\frac{a q k+b}{[q(k+1)-1] q(k+1)} c_{k}
$$

Taking into account 2.5 we find

$$
\begin{gathered}
c_{n}=(-1)^{n} \frac{b(a q+b) \ldots((n-1) a q+b)}{(q-1)(2 q-1) \ldots(n q-1)} \frac{1}{q^{n} n!} \\
=(-1)^{n} \frac{(b / a q)_{n}}{(1-1 / q)_{n}}\left(\frac{a}{q}\right)^{n} \frac{1}{n!}
\end{gathered}
$$

The solution of the Eq. 2.3 with the initial conditions 2.4 is

$$
f(x)=1+\sum_{n=1}^{\infty}(-1)^{n} \frac{(b / a q)_{n}}{(1-1 / q)_{n}}\left(\frac{a}{q}\right)^{n} \frac{x^{n q}}{n!}
$$

i.e. $f(x)$ is the confluent hypergeometric function

$$
f(x)={ }_{1} F_{1}\left(\frac{b}{a q}, 1-\frac{1}{q} ;-\frac{a x^{q}}{q}\right)
$$

The second solution which satisfies

$$
f(0)=0, \quad f^{\prime}(0)=1
$$

is found in the same way and is given by

$$
f(x)=x_{1} F_{1}\left(\frac{a+b}{a q}, \frac{q+1}{q} ;-\frac{a x^{q}}{q}\right)
$$

The solutions 2.6 and 2.7 give the general solution of Eq.2.3 and as function of $q$ have poles at $q= \pm 1 / n+1$, respectively, which accumulate at $q=0$. When $b / a q=-p$, and $a+b / a q=-n$, respectively the solutions reduce to polynomials in $x^{q}$.

### 2.3 Bessel Polynomials

Until now in all the applications of the decomposition method the operator $L$ contained a piece involving the highest-ordered derivative. We want to present an example where such an approach does not work. If we form $L$ from lower-ordered derivatives we can solve the equation.

We consider the equation

$$
x^{2} f^{\prime \prime}(x)+(2 \alpha x+\beta) f^{\prime}(x)-\gamma(\gamma+1) f(x)=0
$$

where $\alpha, \beta$ and $\gamma$ are complex numbers. If $\alpha=1, \beta=2$ and $\gamma=n$ the above equation is the defining equation for the Bessel polynomials [4] [5].

We take $L$ as the first order operator

$$
L=\frac{d}{d x}
$$

The above equation is rewritten as

$$
L f(x)=f^{\prime}(x)=\frac{1}{\beta}\left[\gamma(\gamma+1) f(x)-2 x \alpha f^{\prime}(x)-x^{2} f^{\prime \prime}(x)\right]
$$

In this way we find the Volterra equation

$$
f(x)=f(0)+\frac{1}{\beta} \int_{0}^{x}\left[\gamma(\gamma+1) f(x)-2 x \alpha f^{\prime}(x)-x^{2} f^{\prime \prime}(x)\right] d x
$$

We look for a regular solution of Eq. 2.8 at the origin and take the initial condition

$$
f_{0}=f(0)=1
$$

We get

$$
f_{1}=\frac{\gamma(\gamma+1)}{\beta} \frac{x}{1!}
$$

We make the ansatz

$$
f_{k}=\frac{c_{k}}{\beta^{k}} \frac{x^{k}}{k!}
$$

From the integral equation we find

$$
f_{k+1}=-\frac{c_{k}\left[k^{2}+(2 \alpha-1) k-\gamma(\gamma+1)\right]}{(k+1)!} \frac{x^{k+1}}{\beta^{k+1}}
$$

Now we define

$$
(2 \nu+1)^{2}=(2 \alpha-1)^{2}+4 \gamma(\gamma+1)
$$

and from the above relations one finds the recurrence relation

$$
c_{k+1}=-(k+\alpha+\nu)(k+\alpha-\nu-1) c_{k}
$$

whose solution is

$$
c_{k}=(-1)^{k}(\alpha+\nu)_{k}(\alpha-\nu-1)_{k}
$$

and the solution of Eq. 2.7 is

$$
f(x)={ }_{2} F_{0}\left(\alpha-\nu-1, \alpha+\nu ;-\frac{x}{\beta}\right)
$$

For $\alpha-\nu-1=-n$ we find the generalized Bessel polynomials [4].

## 3 Volterra Integral Equation

Let be a general second-order differential equation

$$
f^{\prime \prime}(x)+a(x) f^{\prime}(x)+b(x) f(x)=h(x)
$$

and let be $\varphi(x) \neq 0$ a solution of the corresponding homogeneous equation.
By applying a variant of the method of variation of parameters we obtain the general solution of Eq. 3.1 as

$$
\begin{gather*}
f(x)=C_{1} \varphi(x)+C_{2} \varphi(x) \int \frac{d x}{E(x) \varphi^{2}(x)}+ \\
\varphi(x) \int \frac{d x}{E(x) \varphi^{2}(x)} \int E(x) \varphi(x) h(x) d x
\end{gather*}
$$

where $E(x)=\exp \int a(x) d x$ and $C_{1}, C_{2}$ are constants [3].
In the above form $L^{-1}$ is defined as an indefinite integral; for every problem we have to transform it into a definite integral according with the solution we are looking for.

We consider 3.2 as the starting point in the formulation of the most general form of the decomposition method. We use Eq. 3.1 to transform a linear homogeneous equation into a pseudo Volterra integral equation. The crucial point is to separate a convenient part in the left hand side of 3.1 such that we can easily find a solution of this part; this separation has to take into account the behavior of the solution in the neighborhood of the point where we are looking for the solution, to improve our chance to obtain the full series in explicit form. The simplest separation is by moving the term $b(x) f(x)$ on the right-hand side, then a solution of the left side alone is $f(x)=1$. We shall give a few examples rather than treat the general case.

Let us come back to the hypergeometric equation 2.1 and write it as

$$
f^{\prime \prime}(x)+\frac{\gamma}{x} f^{\prime}(x)=x f^{\prime \prime}(x)+(\alpha+\beta+1) f^{\prime}(x)+\frac{\alpha \beta}{x} f(x)
$$

We treat the right-hand side of this equation as a formal non-homogeneous term and look for a solution of the homogeneous part, i.e. of the left part.

The solutions are $\varphi_{+}(x)=1$ and $\varphi_{-}(x)=x^{1-\gamma}$ We consider first $\varphi_{+}$and take $C_{1}=1$ and $C_{2}=0$. Since we look for the regular solution at $x=0$ we take $L^{-1}$ as a 2 -fold integration from 0 to $x$ and find from 3.2 the Volterra integral equation

$$
f(x)=1+\int_{0}^{x} \frac{d x}{x^{\gamma}} \int_{0}^{x} x^{\gamma}\left[x f^{\prime \prime}(x)+(\alpha+\beta+1) f^{\prime}(x)+\frac{\alpha \beta}{x} f(x)\right] d x
$$

We solve Eq. 3.3 by iteration taking the zero-order approximation the nonhomogeneous term $f(0)=f_{0}=1$ and find

$$
f_{n+1}=\int_{0}^{x} \frac{d x}{x^{\gamma}} \int_{0}^{x} x^{\gamma}\left[x f_{n}^{\prime \prime}(x)+(\alpha+\beta+1) f_{n}^{\prime}(x)+\frac{\alpha \beta}{x} f_{n}(x)\right] d x
$$

which coincides with the result found in the preceding section.
If we start with the second solution $\varphi_{-}$and take as above $C_{1}=1$ and $C_{2}=0$ we find the integral equation

$$
f(x)=x^{1-\gamma}+x^{1-\gamma} \int_{0}^{x} d x x^{\gamma-2} \int_{0}^{x}\left[x f^{\prime \prime}(x)+(\alpha+\beta+1) f^{\prime}(x)+\frac{\alpha \beta}{x} f(x)\right] d x
$$

We solve it by iteration, but now the zero-order approximation is $f_{0}=x^{1-\gamma}$, and find

$$
f_{1}=x^{1-\gamma} \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{2-\gamma} \frac{x}{1!}
$$

By induction we get the series for the second solution

$$
f(x)=x^{1-\gamma}{ }_{2} F_{1}(\alpha-\gamma+1, \beta-\gamma+1 ; 2-\gamma ; x)
$$

We consider now the equation of the Bessel functions and write it as

$$
f^{\prime \prime}(x)+\frac{1}{x} f^{\prime}(x)+\left(1-\frac{\nu^{2}}{x^{2}}\right) f(x)=0
$$

Its solution cannot be obtained by the decomposition method as originally stated, nor in the more general form of it given by us in Sec. 2. The reason is the singular behavior of the Bessel functions in the neighborhood of the point $x=0$; however its solution can be easily obtained using the form 3.2 . Indeed, we rewrite the Bessel equation in the form

$$
f^{\prime \prime}(x)+\frac{1}{x} f^{\prime}(x)-\frac{\nu^{2}}{x^{2}} f(x)=-f(x)
$$

and observe that the "homogeneous" part has the solutions $\varphi_{ \pm}(x)=x^{ \pm \nu}$. Since the behaviour of the Bessel functions at $x=0$ is that given by $\varphi_{ \pm}$we choose $L^{-1}$ as a 2-fold integration from 0 to $x$.

Let us try the first solution; we choose $C_{1}=2^{-\nu} / \Gamma(\nu+1)$ and $C_{2}=0$ and from 3.2 we get the integral equation

$$
f(x)=\frac{(x / 2)^{\nu}}{\Gamma(\nu+1)}-x^{\nu} \int_{0}^{x} d x x^{-1-2 \nu} \int_{0}^{x} d x x^{1+\nu} f(x)
$$

The initial approximation of the solution is the non-homogeneous term which is no more constant

$$
f_{0}=\frac{(x / 2)^{\nu}}{\Gamma(\nu+1)}
$$

and from the above equation we get

$$
f_{1}=-\frac{(x / 2)^{\nu+2}}{\Gamma(\nu+2)}
$$

and by induction we find the known series.

$$
f(x)=J_{\nu}(x)=\sum_{k=0}^{\infty}(-1)^{k} \frac{(x / 2)^{\nu+2 k}}{k!\Gamma(\nu+k+1)}
$$

If we choose the initial conditions as

$$
C_{1}=0, \quad C_{2}=\frac{2^{-\nu}}{\Gamma(1-\nu)}
$$

proceeding as above we find the second independent solution, $J_{-\nu}(x)$.
The equation 3.2 represents the starting point for obtaining the most general form of the decomposition method. Indeed it can be recovered, in the form given by Adamian, by the choice $a(x)=b(x)=0$ in Eq. 3.1. As a consequence $\varphi(x)=1$ and $E(x) \equiv 1$ and if we look for regular solutions in the neighborhood of $x=03.2$ has the form

$$
f(x)=C_{1}+C_{2} x+\int_{0}^{x} d x \int_{0}^{x} h(x) d x
$$

and by identification

$$
C_{1}=f(0) \quad C_{2}=f^{\prime}(0)
$$

## 4 Conclusion

We have shown that with the necessary modificatons the Adomian's method can be used to obtain the classical results on the special functions. Rather than prescribe a unique form we show that the concrete problem decides how to formulate the method. The clue of this one consists in transforming
a linear differential equation into a pseudo Volterra integral equation whose solution is obtained by the Picard process of succesive approximation.

The method can be obtained by a light modification of the known form of the solution for a linear second order differential equation, the sole difference being the extension of what is ussualy called the non-homogeneous term in the sense that it can include important pieces from the homogeneous part.

Our results can also be seen as a good illustration for the effectivness of the Picard method of succesive approximation.

## References

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