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Fractional Sturm–Liouville eigen-problems: Theory and numerical approximation

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ABSTRACT

We first consider a regular fractional Sturm–Liouville problem of two kinds RFSLP-I and RFSLP-II of order $\nu \in (0, 2)$. The corresponding fractional differential operators in these problems are both of Riemann–Liouville and Caputo type, of the same fractional order $\mu = \nu/2 \in (0, 1)$. We obtain the analytical eigensolutions to RFSLP-I & -II as non-polynomial functions, which we define as Jacobi *poly-fractonomials*. These eigenfunctions are orthogonal with respect to the weight function associated with RFSLP-I & -II. Subsequently, we extend the fractional operators to a new family of singular fractional Sturm–Liouville problems of two kinds, SFSLP-I and SFSLP-II. We show that the primary regular boundary-value problems RFSLP-I & -II are indeed asymptotic cases for the singular counterparts SFSLP-I & -II. Furthermore, we prove that the eigenvalues of the singular problems are real-valued and the corresponding eigenfunctions are orthogonal. In addition, we obtain the eigen-solutions to SFSLP-I & -II analytically, also as non-polynomial functions, hence completing the whole family of the Jacobi poly-fractonomials. In numerical examples, we employ the new poly-fractonomial bases to demonstrate the exponential convergence of the approximation in agreement with the theoretical results.

1. Introduction

The Sturm–Liouville theory has been the keystone for the development of spectral methods and the theory of self-adjoint operators [1]. For many applications, the Sturm–Liouville Problems (SLPs) are studied as boundary-value problems [2]. However, to date mostly integer-order differential operators in SLPs have been used, and such operators do not include any fractional differential operators. Fractional calculus is a theory which unifies and generalizes the notions of integer-order differentiation and integration to any real- or complex-order [3–5].

Over the last decade, it has been demonstrated that many systems in science and engineering can be modeled more accurately by employing fractional-order rather than integer-order derivatives [6–8]. In most of the fractional Sturm–Liouville formulations presented recently, the ordinary derivatives in a traditional Sturm–Liouville problem are replaced with fractional derivatives, and the resulting problems are solved using some numerical schemes such as Adomian decomposition method [9], or fractional differential transform method [10], or alternatively using the method of Haar wavelet operational matrix [11]. However, in such numerical studies, round-off errors and the pseudo-spectra introduced in approximating the infinite-dimensional boundary-value problem as a finite-dimensional eigenvalue problem prohibit computing more than a handful of eigenvalues and eigenfunctions with the desired precision. Furthermore, these papers do not examine the common properties of Fractional Sturm–Liouville Problems (FSLPs) such as orthogonality of the eigenfunctions of the fractional operator in addition to the reality or complexity of the eigensolutions.

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Establishing the aforementioned fundamental properties for FSLPs is very important in establishing proper numerical methods, e.g. the eigensolutions of the RFSLP may be complex [12]. To this end, some results have been recently provided in [13,14], where the fractional character of the problem has been considered through defining a classical Sturm–Liouville operator, extended by the term that includes a sum of the left- and right-sided fractional derivatives. More recently, a Regular Fractional Sturm–Liouville Problem (RFSLP) of two types has been defined in [15], where it has been shown that the eigenvalues of the problem are real, and the eigenfunctions corresponding to distinct eigenvalues are orthogonal. However, the discreteness and simplicity of the eigenvalues have not been addressed. In addition, the spectral properties of a regular FSLP for diffusion operator have been studied in [16] demonstrating that the fractional diffusion operator is self-adjoint. The recent progress in FSLPs is promising for developing new spectral methods for fractional PDEs, however, the eigensolutions have not been obtained explicitly and no numerical approximation results have been published so far.

The main contribution of this paper is to develop a spectral theory for the Regular and Singular Fractional Sturm–Liouville Problems (RFSLP & SFSLP) and demonstrate its utility by constructing explicitly proper bases for numerical approximations of fractional functions. To this end, we first consider a regular problem of two kinds, i.e., RFSLP-I & -II. Then, we obtain the analytical eigensolutions to these problems explicitly for the first time. We show that the explicit eigenvalues to RFSLP-I & -II are real, discrete and simple. In addition, we demonstrate that the corresponding eigenfunctions are of non-polynomial form, called Jacobi *poly-fractonomials*. We also show that these eigenfunctions are orthogonal and dense in $L^2_w[-1,1]$, forming a complete basis in the Hilbert space. We subsequently extend the regular problem to a singular fractional Sturm–Liouville problem again of two kinds SFSLP-I & -II, and prove that the eigenvalues of these singular problems are real and the eigenfunctions corresponding to distinct eigenvalues are orthogonal; these too are computed analytically. We show that the eigensolutions to such singular problems share many fundamental properties with their regular counterparts, with the explicit eigenfunctions of SFSLP-I & -II completing the family of the Jacobi poly-fractonomials. Finally, we complete the spectral theory for the regular and singular FSLPs by analyzing the approximation properties of the eigenfunctions of RFSLPs and SFSLPs, which are employed as basis functions in approximation theory. Our numerical tests verify the theoretical exponential convergence in approximating non-polynomial functions in $L^2_w[-1,1]$. We compare with the standard polynomial basis functions (such as Legendre polynomials) demonstrating the fast exponential convergence of the poly-fractonomial bases.

In the following, we first present some preliminary of fractional calculus in Section 2, and we proceed with the theory on RFSLP and SFSLP in Sections 3 and 4. In Section 5 we present numerical approximations of selected functions and we summarize our results in Section 6.

2. Definitions

Before defining the boundary-value problem, we start with some preliminary definitions of fractional calculus [4]. The left-sided and right-sided Riemann–Liouville integrals of order μ, when $0<\mu<1$, are defined, respectively, as

$$\left({}^{RL}_{x_L}\mathcal{I}^{\mu}_{x} f\right)(x) = \frac{1}{\Gamma(\mu)}\int_{x_L}^{x} \frac{f(s)\,ds}{(x-s)^{1-\mu}}, \quad x > x_L, \tag{1}$$

and

$$\left({}^{RL}_{x}\mathcal{I}^{\mu}_{x_R} f\right)(x) = \frac{1}{\Gamma(\mu)}\int_{x}^{x_R} \frac{f(s)\,ds}{(s-x)^{1-\mu}}, \quad x < x_R, \tag{2}$$

where Γ represents the Euler gamma function. The corresponding inverse operators, i.e., the left-sided and right-sided fractional derivatives of order μ, are then defined based on (1) and (2), as

$$\left({}^{RL}_{x_L}\mathcal{D}^{\mu}_{x} f\right)(x) = \frac{d}{dx}\left({}^{RL}_{x_L}\mathcal{I}^{1-\mu}_{x} f\right)(x) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_{x_L}^{x} \frac{f(s)\,ds}{(x-s)^{\mu}}, \quad x > x_L, \tag{3}$$

and

$$\left({}^{RL}_{x}\mathcal{D}^{\mu}_{x_R} f\right)(x) = \frac{-d}{dx}\left({}^{RL}_{x}\mathcal{I}^{1-\mu}_{x_R} f\right)(x) = \frac{1}{\Gamma(1-\mu)}\left(\frac{-d}{dx}\right)\int_{x}^{x_R} \frac{f(s)\,ds}{(s-x)^{\mu}}, \quad x < x_R. \tag{4}$$

Furthermore, the corresponding left- and right-sided Caputo derivatives of order $\mu \in (0,1)$ are obtained as

$$\left({}^{C}_{x_L}\mathcal{D}^{\mu}_{x} f\right)(x) = \left({}^{RL}_{x_L}\mathcal{I}^{1-\mu}_{x} \frac{df}{dx}\right)(x) = \frac{1}{\Gamma(1-\mu)}\int_{x_L}^{x} \frac{f'(s)\,ds}{(x-s)^{\mu}}, \quad x > x_L, \tag{5}$$

and

$$\left({}^{C}_{x}\mathcal{D}^{\mu}_{x_R} f\right)(x) = \left({}^{RL}_{x}\mathcal{I}^{1-\mu}_{x_R} \frac{-df}{dx}\right)(x) = \frac{1}{\Gamma(1-\mu)} \int_{x_L}^{x} \frac{-f'(s)\,ds}{(x-s)^{\mu}}, \quad x < x_R. \tag{6}$$

The two definitions of the left- and right-sided fractional derivatives of both Riemann–Liouville and Caputo type are linked by the following relationship, which can be derived by a direct calculation

$$\left({}^{RL}_{x_L}\mathcal{D}^{\mu}_{x} f\right)(x) = \frac{f(x_L)}{\Gamma(1-\mu)(x-x_L)^{\mu}} + \left({}^{C}_{x_L}\mathcal{D}^{\mu}_{x} f\right)(x), \tag{7}$$

and

$$\left({}^{RL}_{x}\mathcal{D}^{\mu}_{x_R} f\right)(x) = \frac{f(x_R)}{\Gamma(1-\mu)(x_R-x)^{\mu}} + \left({}^{C}_{x}\mathcal{D}^{\mu}_{x_R} f\right)(x). \tag{8}$$

Moreover, the fractional integration-by-parts for the aforementioned fractional derivatives is obtained as

$$\int_{x_L}^{x_R} f(x)\, {}^{RL}_{x}\mathcal{D}^{\mu}_{x_R} g(x)\, dx = \int_{x_L}^{x_R} g(x)\, {}^{C}_{x_L}\mathcal{D}^{\mu}_{x} f(x)\, dx - f(x)\, {}^{RL}_{x}\mathcal{I}^{\mu}_{x_R} g(x)\Big|_{x=x_L}^{x_R}, \tag{9}$$

and

$$\int_{x_L}^{x_R} f(x)\, {}^{RL}_{x_L}\mathcal{D}^{\mu}_{x} g(x)\, dx = \int_{x_L}^{x_R} g(x)\, {}^{C}_{x}\mathcal{D}^{\mu}_{x_R} f(x)\, dx + f(x)\, {}^{RL}_{x_L}\mathcal{I}^{\mu}_{x} g(x)\Big|_{x=x_L}^{x_R}. \tag{10}$$

Finally, we recall a useful property of the Riemann–Liouville fractional derivatives. Assume that $0 < p < 1$ and $0 < q < 1$ and $f(x_L) = 0$ $x > x_L$, then

$${}^{RL}_{x_L}\mathcal{D}^{p+q}_{x} f(x) = \left({}^{RL}_{x_L}\mathcal{D}^{p}_{x}\right)\left({}^{RL}_{x_L}\mathcal{D}^{q}_{x}\right) f(x) = \left({}^{RL}_{x_L}\mathcal{D}^{q}_{x}\right)\left({}^{RL}_{x_L}\mathcal{D}^{p}_{x}\right) f(x). \tag{11}$$

3. Part I: Regular fractional Sturm–Liouville problems of kind I & II

We consider a regular fractional Sturm–Liouville problem (RFSLP) of order $\nu \in (0,2)$ [15], where the differential part contains both left- and right-sided fractional derivatives, each of order $\mu = \nu/2 \in (0,1)$ as

$${}^{RL}\mathcal{D}^{\mu}\left[p_i(x)\, {}^{C}\mathcal{D}^{\mu} \Phi^{(i)}_{\lambda}(x)\right] + q_i(x)\Phi^{(i)}_{\lambda}(x) + \lambda w_i(x) \Phi^{(i)}_{\lambda}(x) = 0, \quad x \in [x_L, x_R], \tag{12}$$

where $i \in \{1,2\}$, with $i = 1$ denoting the RFSLP of first kind, where ${}^{RL}\mathcal{D}^{\mu} \equiv {}^{RL}_{x}\mathcal{D}^{\mu}_{x_R}$ (i.e., right-sided Riemann–Liouville fractional derivative of order μ) and ${}^{C}\mathcal{D}^{\mu} \equiv {}^{C}_{x_L}\mathcal{D}^{\mu}_{x}$ (i.e., left-sided Caputo fractional derivative of order μ), and $i = 2$ corresponding to the RFSLP of second kind in which ${}^{RL}\mathcal{D}^{\mu} \equiv {}^{RL}_{x_L}\mathcal{D}^{\mu}_{x}$ and ${}^{C}\mathcal{D}^{\mu} \equiv {}^{C}_{x}\mathcal{D}^{\mu}_{x_R}$ (i.e., respectively, left-sided Riemann–Liouville and right-sided Caputo fractional derivative of order μ). In such setting, $\mu \in (0,1)$, $p_i(x) \neq 0$, $w_i(x)$ is a non-negative weight function, and $q_i(x)$ is a potential function. Also, p_i, q_i and w_i are real-valued continuous functions in the interval $[x_L, x_R]$.

The boundary-value problem (12) is subject to the boundary conditions

$$a_1 \Phi^{(i)}_{\lambda}(x_L) + a_2\, {}^{RL}\mathcal{I}^{1-\mu}\left[p_i(x)\, {}^{C}\mathcal{D}^{\mu} \Phi^{(i)}_{\lambda}(x)\right]\Big|_{x=x_L} = 0, \tag{13}$$

$$b_1 \Phi^{(i)}_{\lambda}(x_R) + b_2\, {}^{RL}\mathcal{I}^{1-\mu}\left[p_i(x)\, {}^{C}\mathcal{D}^{\mu} \Phi^{(i)}_{\lambda}(x)\right]\Big|_{x=x_R} = 0, \tag{14}$$

where $a_1^2 + a_2^2 \neq 0$, $b_1^2 + b_2^2 \neq 0$. In this notation, ${}^{RL}\mathcal{I}^{1-\mu} \equiv {}^{RL}_{x}\mathcal{I}^{1-\mu}_{x_R}$ (i.e., right-sided Riemann–Liouville fractional integration of order $1-\mu$) when $i = 1$ for RFSLP of first kind, while, ${}^{RL}\mathcal{I}^{1-\mu} \equiv {}^{RL}_{x_L}\mathcal{I}^{1-\mu}_{x}$ (i.e., left-sided Riemann–Liouville fractional integration of order $1-\mu$) when $i = 2$ for RFSLP of second kind.

The problem of finding the eigenvalues λ such that the boundary-value problems (12)–(14) have non-trivial solutions yields the eigenfunction of the regular fractional Sturm–Liouville eigenvalue problem. The following theorem characterizes the eigensolutions we obtain:

Theorem 3.1. *(See [15].) The eigenvalues of (12) are real, and the eigenfunctions, corresponding to distinct eigenvalues in each problem, are orthogonal with respect to the weight functions $w_i(x)$.*

3.1. Regular boundary-value problem definition

In this study, we shall solve two particular forms of RFSLP (12)–(14) denoted by RFSLP-I when $i=1$ and RFSLP-II when $i=2$ of order $\nu = 2\mu \in (0,2)$, where the potential functions $q_i(x)=0$, in both problems. To this end, the following non-local differential operator is defined

$$\mathcal{L}_i^{\mu} := {}^{RL}\mathcal{D}^{\mu}\big[\mathcal{K}\,{}^{C}\mathcal{D}^{\mu}(\cdot)\big], \tag{15}$$

where $\mathcal{K}$ is constant, and by the notation we introduced, $\mathcal{L}_1^{\mu} := {}^{RL}_{\ x}\mathcal{D}^{\mu}_{x_R}[\mathcal{K}\,{}^{C}_{x_L}\mathcal{D}^{\mu}_{x}(\cdot)]$ in RFSLP-I (i.e., first we take the left-sided μ-th order Caputo derivative of the function multiplied by a constant, and then we take the right-sided Riemann–Liouville derivative of order μ), and for the case of RFSLP-II we reverse the order of the right-sided and left-sided derivative for the inner and outer fractional derivatives in the operator, i.e., $\mathcal{L}_2^{\mu} := {}^{RL}_{x_L}\mathcal{D}^{\mu}_{x}[\mathcal{K}\,{}^{C}_{\ x}\mathcal{D}^{\mu}_{x_R}(\cdot)]$, where $\mu \in (0,1)$. In fact, we have set $p_i(x)=\mathcal{K}$, a continuous non-zero constant function $\forall x \in [-1,1]$. We referred to $\mathcal{K}$ as *stiffness* constant, which yields the regularity character to the boundary-value problem. That being defined, we consider the RFSLP (-I & -II) as

$$\mathcal{L}_i^{\mu}\Phi_{\lambda}^{(i)}(x) + \lambda(1-x)^{-\mu}(1+x)^{-\mu}\Phi_{\lambda}^{(i)}(x) = 0, \quad i \in \{1,2\},\ x \in [-1,1]. \tag{16}$$

We shall solve (16) subject to a homogeneous Dirichlet and a homogeneous fractional *integro-differential* boundary condition to the problems RFSLP-I and RFSLP-II, respectively, as

$$\begin{aligned}&\Phi_{\lambda}^{(1)}(-1) = 0,\\ &{}^{RL}_{\ x}\mathcal{I}^{1-\mu}_{1}\big[\mathcal{K}\,{}^{C}_{-1}\mathcal{D}^{\mu}_{x}\Phi_{\lambda}^{(1)}(x)\big]\big|_{x=+1} = 0,\end{aligned} \tag{17}$$

and

$$\begin{aligned}&\Phi_{\lambda}^{(2)}(+1) = 0,\\ &{}^{RL}_{-1}\mathcal{I}^{1-\mu}_{x}\big[\mathcal{K}\,{}^{C}_{\ x}\mathcal{D}^{\mu}_{1}\Phi_{\lambda}^{(2)}(x)\big]\big|_{x=-1} = 0.\end{aligned} \tag{18}$$

The boundary conditions (17) and (18) are natural in non-local calculus and fractional differential equations, and they are motivated by the fractional integration-by-parts (9) and (10). In fact, the fundamental properties of eigensolutions in the theory of classical Sturm–Liouville problems are connected with the integration-by-parts formula and the choice of the boundary conditions. For instance, the continuity or discreteness of the eigen-spectrum in boundary-value problems is highly dependent on the type of boundary conditions enforced. In the setting chosen here, we shall show that the eigen-spectra of RFSLP-I and RFSLP-II are simple and fully discrete.

3.2. Analytical eigensolutions to RFSLP-I & -II

Here, we obtain the analytical solution $\Phi_{\lambda}^{(i)}(x)$ to RFSLP-I & -II, (16), subject to the homogeneous Dirichlet and integro-differential boundary conditions (17) and (18). Before that, we recall the following lemmas for the standard Jacobi polynomials $P_n^{\alpha,\beta}$:

Lemma 3.2. *(See [17].) For $\mu > 0$, $\alpha > -1$, $\beta > -1$, and $\forall x \in [-1,1]$*

$$(1+x)^{\beta+\mu}\frac{P_n^{\alpha-\mu,\beta+\mu}(x)}{P_n^{\alpha-\mu,\beta+\mu}(-1)} = \frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)P_n^{\alpha,\beta}(-1)}\int_{-1}^{x}\frac{(1+s)^{\beta}P_n^{\alpha,\beta}(s)}{(x-s)^{1-\mu}}\,ds. \tag{19}$$

By the left-sided Riemann–Liouville integral (1) and evaluating the special end-values $P_n^{\alpha-\mu,\beta+\mu}(-1)$ and $P_n^{\alpha,\beta}(-1)$, we can re-write (19) as

$${}^{RL}_{-1}\mathcal{I}^{\mu}_{x}\big\{(1+x)^{\beta}P_n^{\alpha,\beta}(x)\big\} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\mu+1)}(1+x)^{\beta+\mu}P_n^{\alpha-\mu,\beta+\mu}(x). \tag{20}$$

Lemma 3.2 can be reduced to the case when $\alpha = +\mu$ and $\beta = -\mu$ as

$${}^{RL}_{-1}\mathcal{I}^{\mu}_{x}\big\{(1+x)^{-\mu}P_n^{\mu,-\mu}(x)\big\} = \frac{\Gamma(n-\mu+1)}{\Gamma(n+1)}P_n(x), \tag{21}$$

where $P_n(x) = P_n^{0,0}(x)$ represents the Legendre polynomial of degree n. On the other hand, we can set $\alpha=\beta=0$ in (20) and take the fractional derivative ${}^{RL}_{-1}\mathcal{D}^{\mu}_{x}$ on both sides of (20) to obtain

$${}^{RL}_{-1}\mathcal{D}^{\mu}_{x}\big\{(1+x)^{\mu}P_n^{-\mu,\mu}\big\} = \frac{\Gamma(n+\mu+1)}{\Gamma(n+1)}P_n(x). \tag{22}$$

Lemma 3.3. *(See [17].) For $\mu > 0$, $\alpha > -1$, $\beta > -1$, and $\forall x \in [-1, 1]$*

$$(1-x)^{\alpha+\mu} \frac{P_n^{\alpha+\mu,\beta-\mu}(x)}{P_n^{\alpha+\mu,\beta-\mu}(+1)} = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)P_n^{\alpha,\beta}(+1)} \int_x^1 \frac{(1-s)^{\alpha} P_n^{\alpha,\beta}(s)}{(s-x)^{1-\mu}} ds. \tag{23}$$

By the right-sided Riemann–Liouville integral (2) and evaluating the special end-values $P_n^{\alpha-\mu,\beta+\mu}(+1)$ and $P_n^{\alpha,\beta}(+1)$, we can re-write (23) as

$${}^{RL}_{x}\mathcal{I}^{\mu}_{1}\big\{(1-x)^{\alpha} P_n^{\alpha,\beta}(x)\big\} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\mu+1)}(1-x)^{\alpha+\mu} P_n^{\alpha+\mu,\beta-\mu}(x). \tag{24}$$

Similarly, Lemma 3.3 for the case $\alpha = -\mu$ and $\beta = +\mu$ leads to

$${}^{RL}_{x}\mathcal{I}^{\mu}_{1}\big\{(1-x)^{-\mu} P_n^{-\mu,+\mu}(x)\big\} = \frac{\Gamma(n-\mu+1)}{\Gamma(n+1)} P_n(x). \tag{25}$$

On the other hand, one can set $\alpha = \beta = 0$ in (24) and take the fractional derivative ${}^{RL}_{x}\mathcal{D}^{\mu}_{1}$ on both sides of (24) to obtain

$$f\,{}^{RL}_{x}\mathcal{D}^{\mu}_{1}\big\{(1-x)^{\mu} P_n^{\mu,-\mu}\big\} = \frac{\Gamma(n+\mu+1)}{\Gamma(n+1)} P_n(x). \tag{26}$$

Relations (21), (22), (25) and (26) are the key to proving the following theorem.

Theorem 3.4. *The exact eigenfunctions to (16), when $i = 1$, i.e., RFSLP-I, subject to (17) are given as*

$$\Phi_n^{(1)}(x) = (1+x)^{\mu} P_{n-1}^{-\mu,\mu}(x), \quad \forall n \geqslant 1, \tag{27}$$

and the corresponding distinct eigenvalues are

$$\lambda_n^{(1)} = -\frac{\mathcal{K}\Gamma(n+\mu)}{\Gamma(n-\mu)}, \quad \forall n \geqslant 1. \tag{28}$$

Moreover, the exact eigenfunctions to (16), when $i = 2$, i.e., RFSLP-II subject to (18), are given as

$$\Phi_n^{(2)}(x) = (1-x)^{\mu} P_{n-1}^{\mu,-\mu}(x), \quad \forall n \geqslant 1 \tag{29}$$

where the corresponding distinct eigenvalues are given as

$$\lambda_n^{(2)} = \lambda_n^{(1)} = -\frac{\mathcal{K}\Gamma(n+\mu)}{\Gamma(n-\mu)}, \quad \forall n \geqslant 1. \tag{30}$$

Proof. We split the proof into three parts.

Part a: First, we prove (27) and (28). From (27), it is clear that $\Phi_n^{(1)}(-1) = 0$. To check the other boundary condition, since $\Phi_n^{(1)}(-1) = 0$, by property (7), we could replace ${}^{C}_{-1}\mathcal{D}^{\mu}_{x}$ by ${}^{RL}_{-1}\mathcal{D}^{\mu}_{x}$, hence,

$$\begin{aligned}
&\big\{{}^{RL}_{x}\mathcal{I}^{1-\mu}_{+1}\big[\mathcal{K}\,{}^{C}_{-1}\mathcal{D}^{\mu}_{x}\Phi_n^{(1)}(x)\big]\big\}_{x=+1} \\
&\quad = \big\{{}^{RL}_{x}\mathcal{I}^{1-\mu}_{+1}\big[\mathcal{K}\,{}^{RL}_{-1}\mathcal{D}^{\mu}_{x}\Phi_n^{(1)}(x)\big]\big\}_{x=+1} \\
&\quad = \big\{{}^{RL}_{x}\mathcal{I}^{1-\mu}_{+1}\big[\mathcal{K}\,{}^{RL}_{-1}\mathcal{D}^{\mu}_{x}\big((1+x)^{\mu} P_{n-1}^{-\mu,\mu}(x)\big)\big]\big\}_{x=+1}
\end{aligned}$$

(and by carrying out the fractional RL derivative in the bracket using (22))

$$= \left\{{}^{RL}_{x}\mathcal{I}^{1-\mu}_{+1}\left[\mathcal{K}\frac{\Gamma(n+\mu)}{\Gamma(n)}P_{n-1}(x)\right]\right\}_{x=+1} = \mathcal{K}\frac{\Gamma(n+\mu)}{\Gamma(n)}\big\{{}^{RL}_{x}\mathcal{I}^{1-\mu}_{+1}\big[Pf_{n-1}(x)\big]\big\}_{x=+1}.$$

By working out the fractional integration using (24) (when $\alpha = \beta = 0$), we obtain

$$\mathcal{K}\big\{(1-x)^{\mu} P_{n-1}^{\mu,-\mu}\big\}_{x=+1} = 0.$$

Now, we need to show that (27) indeed satisfies (16), when $i = 1$, with the eigenvalues (28). First, we take the fractional integration of order μ on both sides of (16) taking $i = 1$,

$$\mathcal{K}\,{}^{C}_{-1}\mathcal{D}^{\mu}_{x}\Phi_n^{(1)}(x) = -\lambda\,{}^{RL}_{x}\mathcal{I}^{\mu}_{+1}\big\{(1-x)^{-\mu}(1+x)^{-\mu}\Phi_n^{(1)}(x)\big\}.$$

Substituting (27) and replacing the Caputo derivative by the Riemann–Liouville one, thanks to (7), we obtain

$$\mathcal{K}\,{}_{-1}^{RL}\mathcal{D}_x^{\mu}\big[(1+x)^{\mu}P_{n-1}^{-\mu,\mu}(x)\big] = -\lambda\,{}_x^{RL}\mathcal{I}_{+1}^{\mu}\big\{(1-x)^{-\mu}P_{n-1}^{-\mu,\mu}(x)\big\}.$$

Finally, the fractional derivative on the left-hand side and the fractional integration on the right-hand side are worked out using (22) and (25), respectively, as

$$\mathcal{K}\frac{\Gamma(n+\mu)}{\Gamma(n)}P_{n-1}(x) = -\lambda\frac{\Gamma(n-\mu)}{\Gamma(n)}P_{n-1}(x).$$

Since, $P_{n-1}^{\alpha+1,-\beta-1}(x)$ is non-zero almost everywhere in $[-1,1]$, we can cancel this term out from both sides and get

$$\lambda \equiv \lambda_n^{(1)} = -\frac{\mathcal{K}\Gamma(n+\mu)}{\Gamma(n-\mu)}, \quad \forall n \geqslant 1,$$

which shows that the eigenvalues of RFSLP-I are real-valued and discrete. In fact, this result agrees with Theorems 3.1. Moreover, the orthogonality of the eigenfunctions (27) with respect to $w_1(x) = (1-x)^{-\mu}(1+x)^{-\mu}$ is shown as

$$\begin{aligned}\int_{-1}^{1} w_1(x)\Phi_k^{(1)}(x)\Phi_j^{(1)}(x)\,dx &= \int_{-1}^{1} w_1(x)\big[(1+x)^{\mu}\big]^2 P_{k-1}^{-\mu,\mu}(x)P_{j-1}^{-\mu,\,\mu}(x)\,dx\\ &= \int_{-1}^{1}(1-x)^{-\mu}(1+x)^{\mu}P_{k-1}^{-\mu,\mu}(x)P_{j-1}^{-\mu,\mu}(x)\,dx\\ &= C_k^{-\mu,\mu}\delta_{kj},\end{aligned}$$

where $C_k^{-\mu,\mu}$ denotes the orthogonality constant of the family of Jacobi polynomials with parameters ${}^{-\mu,\mu}$.

Part b: The proof of the eigen-solution to RFSLP-II, (29) and (30), follows the same steps as in Part a. It is clear that $\Phi_n^{(2)}(1) = 0$. To check the other boundary condition in (18), since $\Phi_n^{(2)}(1) = 0$, by (8), we can replace ${}_x^{C}\mathcal{D}_1^{\mu}$ by ${}_x^{RL}\mathcal{D}_1^{\mu}$; hence, by substituting in (29), and working out the middle fractional derivative using (26),

$$\begin{aligned}&\big\{{}_x^{RL}\mathcal{I}_{+1}^{1-\mu}\big[\mathcal{K}\,{}_x^{C}\mathcal{D}_1^{\mu}\Phi_n^{(1)}(x)\big]\big\}_{x=-1}\\ &\quad= \mathcal{K}\frac{\Gamma(n)}{\Gamma(n+\mu)}\big\{{}_{-1}^{RL}\mathcal{I}_x^{1-\mu}\big[P_{n-1}(x)\big]\big\}_{x=-1},\end{aligned}$$

and by working out the fractional integration using (24) (when $\alpha = \beta = 0$), we obtain

$$\mathcal{K}\big\{(1+x)^{\mu}P_{n-1}^{-\mu,+\mu}\big\}_{x=-1} = 0.$$

To check if (29) satisfies (16), when $i = 2$, we can substitute (29) into (16) and carry out the fractional integration of order μ on both sides using (20). Then, by working out the fractional derivative on the left-hand side using (25) we verify that (29) satisfies the boundary-value problem, provided that (30) are the real-values distinct eigenvalues of RFSLP-II.

Finally, the orthogonality of the eigenfunctions (29) with respect to $w_2(x) = (1-x)^{-\mu}(1+x)^{-\mu}$ is shown as

$$\int_{-1}^{1} w_2(x)\Phi_k^{(2)}(x)\Phi_j^{(2)}(x)\,dx = \int_{-1}^{1}(1-x)^{\mu}(1+x)^{-\mu}P_{k-1}^{\mu,-\mu}(x)P_{j-1}^{\mu,-\mu}(x)\,dx = C_k^{\mu,-\mu}\delta_{kj},$$

where $C_k^{\mu,-\mu}$ represents the orthogonality constant of the family of Jacobi polynomials with parameters ${}^{\mu,-\mu}$.

Part c: It is left to prove that the set $\{\Phi_n^{(i)}(x)\colon\ n = 1,2,\ldots\}$ forms a basis for the infinite-dimensional Hilbert space $L_w^2[-1,1]$, and $\lambda_n^{(i)}$, the corresponding eigenvalue for each n, is simple. Let $f(x) \in L_w^2[-1,1]$ and then clearly $g(x) = (1\pm x)^{-\mu}f(x) \in L_w^2[-1,1]$, as well when $\mu \in (0,1)$. Hence

$$\begin{aligned}&\Big\|\sum_{n=1}^{N} a_n\Phi_n^{(i)}(x) - f(x)\Big\|_{L_w^2[-1,1]}\\ &\quad= \Big\|\sum_{n=1}^{N} a_n(1\pm x)^{\mu}P_{n-1}^{\mp\mu,\pm\mu}(x) - f(x)\Big\|_{L_w^2[-1,1]}\\ &\quad= \Big\|(1\pm x)^{\mu}\Big(\sum_{n=1}^{N} a_n P_{n-1}^{\mp\mu,\pm\mu}(x) - (1\pm x)^{-\mu}f(x)\Big)\Big\|_{L_w^2[-1,1]}\end{aligned}$$

$$= \left\| (1 \pm x)^{\mu} \left(\sum_{n=1}^{N} a_n P_{n-1}^{\mp\mu,\pm\mu}(x) - g(x) \right) \right\|_{L^2_w[-1,1]} \quad \text{(by Cauchy–Schwartz)}$$
$$\leqslant \left\| (1 \pm x)^{\mu} \right\|_{L^2_w[-1,1]} \left\| \sum_{n=1}^{N} a_n P_{n-1}^{\mp\mu,\pm\mu}(x) - g(x) \right\|_{L^2_w[-1,1]}$$
$$\leqslant c \left\| \sum_{n=1}^{N} a_n P_{n-1}^{\mp\mu,\pm\mu}(x) - g(x) \right\|_{L^2_w[-1,1]},$$

where the upper signs are corresponding to RFSLP-I, $i = 1$, and the lower signs are corresponding to the case $i = 2$, i.e., RFSLP-II. Hence,

$$\lim_{N\to\infty} \left\| \sum_{n=1}^{N} a_n \Phi_n^{(i)}(x) - f(x) \right\|_{L^2_w[-1,1]} \leqslant \lim_{N\to\infty} c \left\| \sum_{n=1}^{N} a_n P_{n-1}^{\mp\mu,\pm\mu}(x) - g(x) \right\|_{L^2_w[-1,1]} = 0, \tag{31}$$

by Weierstrass theorem. Therefore, $\sum_{n=1}^{N} a_n \Phi_n^{(i)}(x) \xrightarrow{L^2_w} f(x)$, implying that $\{\Phi_n^{(i)}(x)\colon\ n = 1, 2, \ldots\}$ is dense in the Hilbert space and it forms a basis for $L^2_w[-1, 1]$.

To show the simplicity of the eigenvalues, assume that corresponding to the eigenvalue $\lambda_j^{(i)}$, there exits another eigenfunction $\Phi_j^{(i)*}(x) \in L^2_w[-1, 1]$ in addition to $\Phi_j^{(i)}(x)$, which is by Theorem 3.1 orthogonal to the rest of the eigenfunctions $\Phi_n^{(i)}(x)$, $n \neq j$. By the density of the eigenfunctions set, i.e., (31), we can represent $\Phi_j^{(i)*}(x)$ as

$$\Phi_j^{(i)*}(x) = \sum_{n=1}^{\infty} a_n \Phi_n^{(i)}(x). \tag{32}$$

Now, by multiplying both sides by $\Phi_k^{(i)}(x)$, $k = 1, 2, \ldots$ and $k \neq j$, and integrating with respect to the weight function $w(x)$ we obtain

$$\int_{-1}^{1} w(x)\Phi_j^{(i)*}(x)\Phi_k^{(i)}(x)\,dx = \sum_{n=1}^{\infty} a_n \int_{-1}^{1} w(x)\Phi_n^{(i)}(x)\Phi_k^{(i)}(x)\,dx = a_k C_k \neq 0, \tag{33}$$

which contradicts to Theorem 3.1. Therefore, the eigenvalues $\lambda_n^{(i)}$ are simple, and this completes the proof. □

The growth of the magnitude of the eigenvalues of RFSLP-I & -II, $|\lambda_n^i|$, $i \in \{1, 2\}$, is plotted in Fig. 1, corresponding to three values of $\mu = 0.35$, $\mu = 0.5$, and $\mu = 0.99$. We observe that there are two growth modes, depending on either $\mu \in (0, 1/2)$, where a sublinear growth in $|\lambda_n^1| = |\lambda_n^2|$ is observed, or, $\mu \in (1/2, 1)$, where a superlinear–subquadratic growth mode is noticed. The case $\mu = 1/2$ leads to an exactly linear growth mode.

In order to visually get more sense of how the eigensolutions look like, in Fig. 2 we plot the eigenfunctions of RFSLP-I, $\Phi_n^{(1)}(x)$ of different orders and corresponding to different values of μ used in Fig. 1. In each plot we compare the eigensolutions with the corresponding standard Jacobi polynomials $P_n^{-\mu,\mu}(x)$. In a similar fashion, we plot the eigenfunctions of RFSLP-II, $\Phi_n^{(2)}(x)$, of different orders compared to $P_n^{+\mu,-\mu}(x)$ in Fig. 3.

So far, the eigenfunctions have been defined in the interval $[-1, 1]$. The following lemma provides a useful shifted definition of the $\Phi_n^{(i)}$, which is not only more convenient to work with but also helps exploit some interesting properties.

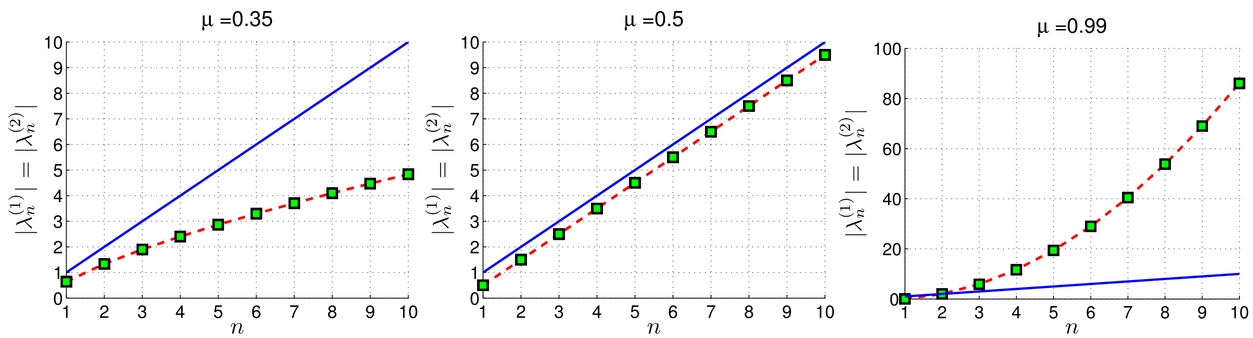

Fig. 1. Magnitude of the eigenvalues of RFSLP-I and RFSLP-II, $|\lambda_n^{(1)}| = |\lambda_n^{(2)}|$, versus n, corresponding to $\mu = 0.35$, left: sublinear growth, $\mu = 0.5$; middle: linear growth, and $\mu = 0.99$; right: superlinear–subquadratic growth. The blue line denotes the linear growth.

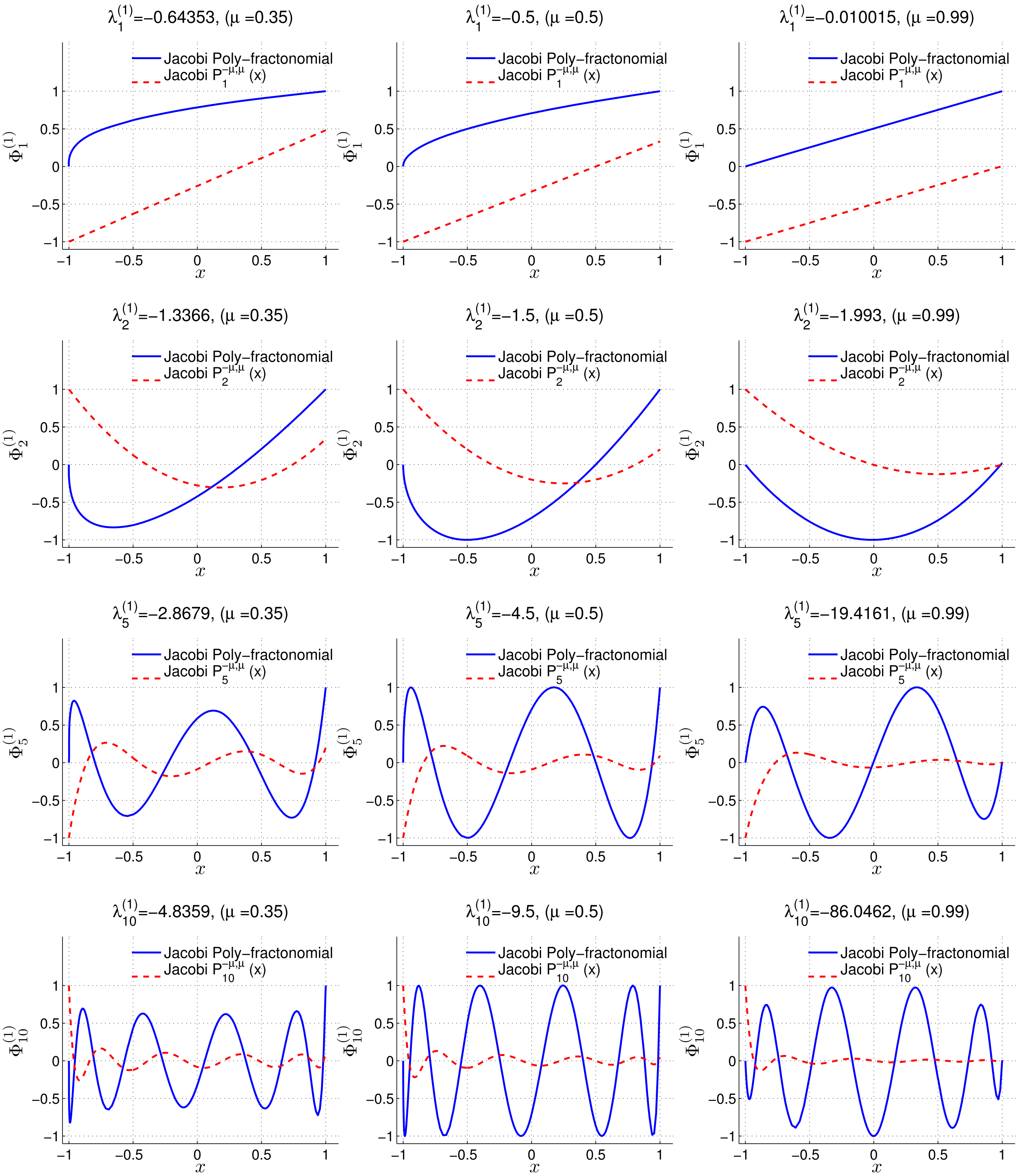

Fig. 2. Eigenfunctions of RFSLP-I, $\Phi_n^{(1)}$, versus x, for $n = 1$ (first row), $n = 2$ (second row), $n = 5$ (third row), and $n = 10$ (last row), corresponding to the fractional order $\mu = \nu/2 = 0.35$ (left column), $\mu = \nu/2 = 0.5$ (middle column), and $\mu = \nu/2 = 0.99$ (right column).

Lemma 3.5. *The shifted eigensolutions to RFSLP-I & -II, denoted by $\widetilde{\Phi}_n^{(i)}(t)$, $i \in \{1, 2\}$, are given by*

$$\widetilde{\Phi}_n^{(i)}(t) = 2^{\mu} \sum_{j=0}^{n-1} (-1)^{n+j-1} \binom{n-1+j}{j} \binom{n-1+(-1)^{i+1}\mu}{n-1-j} t^{j+\mu}, \tag{34}$$

where $t \in [0, 1]$ in the mapped domain, in case of RFSLP-I, and $t \in [-1, 0]$ in RFSLP-II.

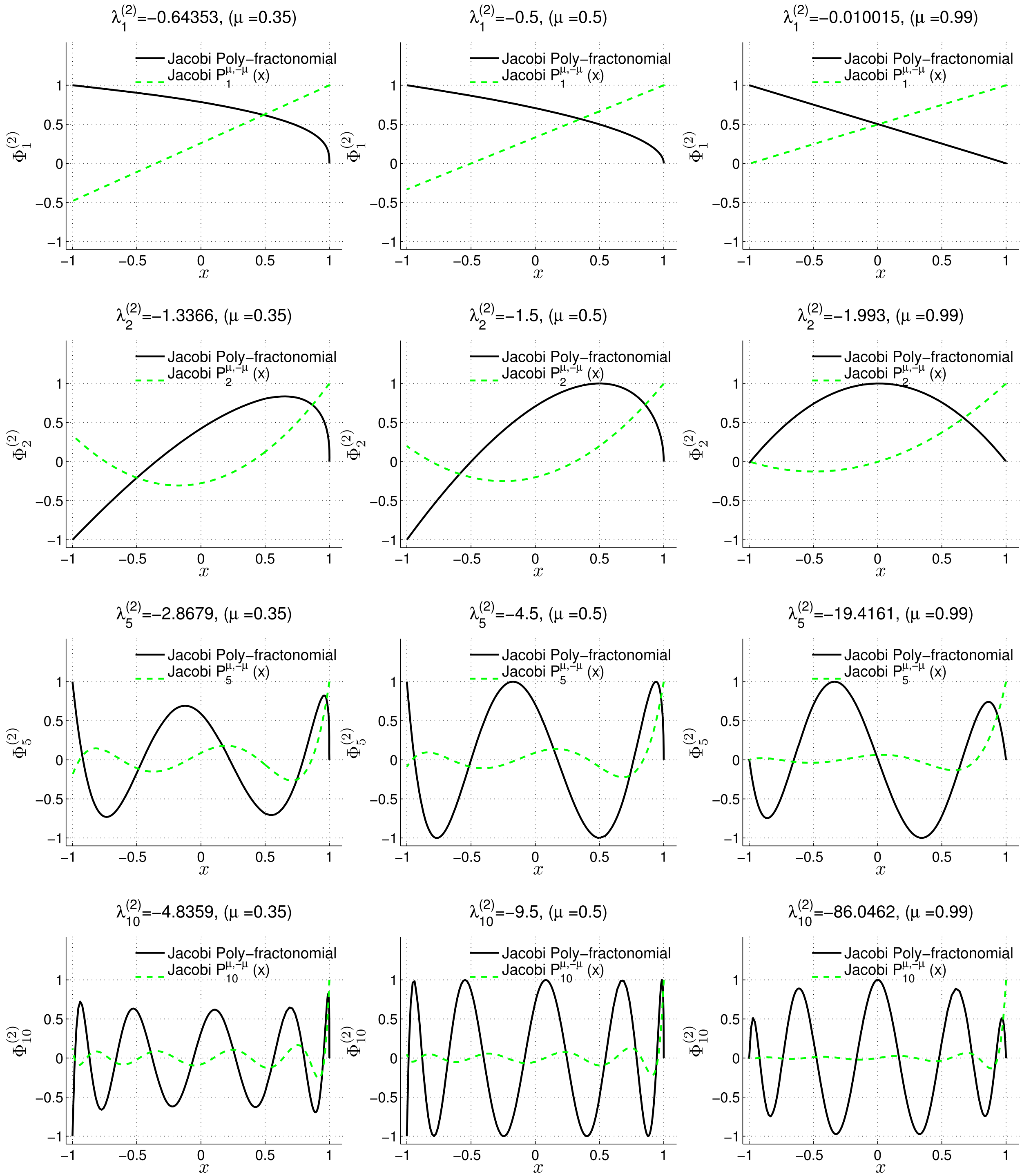

Fig. 3. Eigenfunctions of RFSLP-II, $\Phi_n^{(2)}$, versus x, for $n = 1$ (first row), $n = 2$ (second row), $n = 5$ (third row), and $n = 10$ (last row), corresponding to the fractional order $\mu = \nu/2 = 0.35$ (left column), $\mu = \nu/2 = 0.5$ (middle column), and $\mu = \nu/2 = 0.99$ (right column).

Proof. We first obtain the shifted RFSLP-I by performing an affine mapping from interval $[-1, 1]$ to $[0, 1]$. To do so, we recall the power expansion of the Jacobi polynomial $P_n^{\alpha,\beta}(x)$ as

$$P_n^{\alpha,\beta}(x) = \sum_{j=0}^{n} \binom{n+\alpha+\beta+j}{j} \binom{n+\alpha}{n-j} \left(\frac{x-1}{2}\right)^j, \quad x \in [-1, 1] \tag{35}$$

and from the properties of the Jacobi polynomials we have

$$P_n^{\alpha,\beta}(-x) = (-1)^n P_n^{\beta,\alpha}(x). \tag{36}$$

We obtain the shifted eigensolution $\widetilde{\Phi}_n^{(1)}(t)$ utilizing (35) and (36) in (27) and performing the change of variable $x = 2t - 1$ as

$$\widetilde{\Phi}_n^{(1)}(t) = 2^{\mu} \sum_{j=0}^{n-1} (-1)^{n+j-1} \binom{n-1+j}{j} \binom{n-1+\mu}{n-1-j} t^{j+\mu}. \tag{37}$$

In order to obtain the shifted $\widetilde{\Phi}_n^{(2)}(t)$, we follow the same steps, except that this time we do the change of variable $x = -2t + 1$, which maps $[-1, 1]$ to $[-1, 0]$. □

Definition 3.6. A *fractonomial* is defined as a function $f : \mathbb{C} \to \mathbb{C}$ of non-integer power, denoted as $t^{k+\mu}$, where $k \in \mathbb{Z}^+$ and $\mu \in (0, 1)$, in which the power can be represented as a sum of an integer and non-integer number. Moreover, denoted by $\mathcal{F}_e^{n+\mu}$ is the *fractal expansion set*, which is defined as the set of all fractonomials of order less than or equal $n + \mu$ as

$$\mathcal{F}_e^{n+\mu} = span\{t^{k+\mu}:\ \mu \in (0,1) \text{ and } k = 0, 1, \ldots, n\}. \tag{38}$$

Remark 3.7. All fractonomials are zero-valued at $t = 0$. Moreover, asymptotically, when $\mu \to 0$, a fractonomial of order $n+\mu$ approaches the monomial t^n.

Definition 3.8. A *poly-fractonomial* of order $n + \mu < \infty$, $n \in \{0, 1, 2, \ldots, N < \infty\}$, and $\mu \in (0, 1)$, is defined as a linear combination of the elements in the fractal expansion set $\mathcal{F}_e^{n+\mu}$, as

$$F_{n+\mu}(t) = a_0 t^{\mu} + a_1 t^{1+\mu} + \cdots + a_n t^{n+\mu},$$

where $a_j \in \mathbb{C}$, $j \in \{0, 1, \ldots, n\}$ are constants. Moreover, denoted by $\mathbb{F}^{n+\mu}$ is the space of all poly-fractonomials up to order $n + \mu$. By Remark 3.7, all elements in $\mathbb{F}^{n+\mu}$ asymptotically approach the corresponding standard polynomial of order n with coefficients a_j.

Remark 3.9. It is observed that $\mathbb{F}^{n+\mu} \subset L^2_w$ since $\mu \in (0, 1)$, and hence, all poly-fractonomials in $\mathbb{F}^{n+\mu}$ can be represented as an infinite sum in terms of the shifted eigenfunctions of RFSLP-I & -II. It is true by the density of the eigenfunction in L^2_w, shown in Part c of the proof in Theorem 3.4.

Lemma 3.10. *Any fractional Caputo derivative of order $\mu \in (0, 1)$ of all polynomials up to degree N lies in the space of poly-fractonomials $\mathbb{F}^{n+\widetilde{\mu}}$, where $n = N - 1$, and $\widetilde{\mu} = 1 - \mu \in (0, 1)$.*

Proof. Let $f(t) = \sum_{j=0}^{N} a_j t^j$ be an arbitrary polynomial of order N, i.e., $a_N \neq 0$. From [4] and for $\mu \in (0, 1)$, we have

$${}^C_0\mathcal{D}_t^{\mu} t^k = \begin{cases} 0, & k < \mu, \\ \frac{\Gamma(k+1)}{\Gamma(k+1-\mu)} t^{k-\mu}, & 0 < \mu \leqslant k. \end{cases} \tag{39}$$

Hence, by (39),

$${}^C_0\mathcal{D}_t^{\mu} f(t) = \sum_{j=0}^{N} a_j\, {}^C_0\mathcal{D}_t^{\mu} t^j = \sum_{j=1}^{N} a_j \frac{\Gamma(j+1)}{\Gamma(j+1-\mu)} t^{j-\mu} = \sum_{j=1}^{N} b_j t^{j-\mu}, \tag{40}$$

where $b_j = \frac{\Gamma(j+1)}{\Gamma(j+1-\mu)} a_j$. Taking $n = N - 1$ and $\widetilde{\mu} = 1 - \mu \in (0, 1)$, and the fact that $b_N = \frac{\Gamma(j+1)}{\Gamma(j+1-\mu)} a_N \neq 0$ completes the proof. □

Theorem 3.11. *The shifted eigensolutions to (16), $\widetilde{\Phi}_n^{(i)}(t)$, $n \in \mathbb{N}$ and $n < \infty$, form a complete hierarchical basis for the finite-dimensional space of poly-fractonomials $\mathbb{F}_{n-1+\mu}$, where $\mu \in (0, 1)$.*

Proof. From (34), it is clear that

$$dim\, \mathbb{F}_{n-1+\mu} = dim\{\widetilde{\Phi}_k^{(i)},\ k \in \{1, 2, \ldots, n\}\}. \tag{41}$$

Moreover, we can re-write (34) as

$$\mathbb{T}\vec{t} = \vec{\Phi}^{(i)}, \tag{42}$$

where

$$\vec{t} = \begin{pmatrix} t^{\mu} \\ t^{1+\mu} \\ \vdots \\ t^{n-1+\mu} \end{pmatrix} \quad \text{and} \quad \vec{\Phi}^{(1)} = \begin{pmatrix} \Phi_1^{(i)}(t) \\ \Phi_2^{(i)}(t) \\ \vdots \\ \Phi_n^{(i)}(t) \end{pmatrix},$$

and finally, $\mathbb{T} = \{T_{jk}\}_{j,k=1}^{n}$ is an $n \times n$ matrix obtained as

$$\mathbb{T} = \{T_{jk}\}_{j,k=1}^{n} = (-1)^{k+j-1} \binom{k-1+j}{j} \binom{k-1+(-1)^{i+1}\mu}{k-1-j},$$

which is a *lower-triangular* matrix. Thanks to the orthogonality of the $\Phi_n^{(i)}$, the eigenfunctions are linearly independent, therefore, the matrix $\mathbb{T}$ is invertible. Let $\mathcal{T} = \mathbb{T}^{-1}$, which is also lower triangular. Hence,

$$\vec{t} = \mathcal{T}\vec{\Phi}^{(i)}. \tag{43}$$

In other words, each element in the poly-fractonomial space $\mathbb{F}_{n-1+\mu}$, say $t^{m+\mu}$, $0 \leqslant m \leqslant n-1$, can be uniquely represented through the following expansion

$$t^{m+\mu} = \sum_{j=1}^{n} c_j \Phi_j^{(i)}(t) = \sum_{j=1}^{n} \{\mathcal{T}_{mj}\} \Phi_j^{(i)}(t) = \sum_{j=1}^{m} \{\mathcal{T}_{mj}\} \Phi_j^{(i)}(t), \tag{44}$$

where the last equality holds since $\mathcal{T}$ is a lower-triangular matrix. As seen in (38), the fractal expansion set $\mathcal{F}_e^{n+\mu} \subset \mathcal{F}_e^{n+1+\mu}$, which indicates that the shifted eigen-solutions $\widetilde{\Phi}_n^{(i)}$ form a *hierarchical* expansion basis set. □

3.3. Properties of the eigensolution to RFSLP-I & -II

Next, we list a number of properties of the solutions to RFSLP-I & -II (16):

- **Non-polynomial nature**:
  From $\Phi_n^{(i)}(x)$ shown in (27) when $i=1$, and in (29) corresponding to $i=2$, it is understood that the eigenfunctions exhibit a non-polynomial behavior, thanks to the multiplier $(1 \pm x)^{\mu}$ of fractional power. Hence, to distinguish them from the standard Jacobi polynomials, we refer to $\Phi_n^{(i)}(x)$ as *Jacobi poly-fractonomial* of order $n+\mu$.
- **Asymptotic eigenvalues $\lambda_n^{(i)}$**:
  The growth in the magnitude of eigenvalues in RFSLP with n is dependent on the fractional derivative order μ, as shown in (30). Since $\mu \in (0,1)$, there are two modes of growth in the magnitude of $\lambda_n^{(i)}$, the *sublinear* mode corresponding to $0 < \mu < 1/2$, and *superlinear–subquadratic* mode which corresponds to $1/2 < \mu < 1$. Particularly, when $\mu = 1/2$, the eigenvalues grow linearly with n. Hence, the asymptotic values are summarized as

$$\left|\lambda_n^{(i)}\right| = \begin{cases} \mathcal{K}n^2, & \mu \to 1, \\ \mathcal{K}n, & \mu \to 1/2, \\ \mathcal{K}, & \mu \to 0. \end{cases} \tag{45}$$

- **Recurrence relations**:
  Thanks to the hierarchical property of the eigenfunctions $\Phi_n^{(i)}$, we obtain the following recurrence relations:

$$\begin{aligned} &\Phi_1^{(i)}(x) = (1 \pm x)^{\mu}, \\ &\Phi_2^{(i)}(x) = (1 \pm x)^{\mu}(x \mp \mu), \\ &\quad\vdots \\ &a_n \Phi_{n+1}^{(i)}(x) = b_n x \Phi_n^{(i)}(x) - c_n \Phi_{n-1}^{(i)}(x), \\ &a_n = 4n^2(n-1), \\ &b_n = 2n(2n-1)(2n-2), \\ &c_n = 4n(n-1 \mp \mu)(n-1 \pm \mu), \end{aligned} \tag{46}$$

  where the upper signs correspond to $i=1$, solution to RFSLP-I, and the lower signs correspond to RFSLP-II when $i=2$.

- **Orthogonality**:

$$\int_{-1}^{1} (1-x)^{-\mu}(1+x)^{-\mu}\Phi_k^{(i)}(x)\Phi_m^{(i)}(x)\,dx = \int_{-1}^{1}(1-x)^{\alpha_i}(1+x)^{\beta_i}P_{k-1}^{\alpha_i,\beta_i}(x)P_{m-1}^{\alpha_i,\beta_i}(x)\,dx = \mathcal{J}_k^{\alpha_i,\beta_i}\delta_{kj}, \tag{47}$$

$$\mathcal{J}_k^{\alpha_i,\beta_i} = \frac{2}{2k-1}\frac{\Gamma(k+\alpha_i)\Gamma(k+\beta_i)}{(k-1)!\Gamma(k)}, \tag{48}$$

where $(\alpha_1,\beta_1)=(-\mu,\mu)$ and $(\alpha_2,\beta_2)=(\mu,-\mu)$.

- **Fractional derivatives**:

$$^{RL}_{-1}\mathcal{D}_x^{\mu}\Phi_n^{(1)} = {}^{C}_{-1}\mathcal{D}_x^{\mu}\Phi_n^{(1)} = {}^{RL}_{x}\mathcal{D}_1^{\mu}\Phi_n^{(2)} = {}^{C}_{x}\mathcal{D}_1^{\mu}\Phi_n^{(2)} = \frac{\Gamma(n+\mu)}{\Gamma(n)}P_{n-1}(x), \tag{49}$$

where $P_{n-1}(x)$ denotes that standard Legendre polynomial of order $n-1$.

- **Orthogonality of the fractional derivatives**:

$$\int_{-1}^{1}\mathcal{D}^{\mu}\Phi_k^{(i)}\mathcal{D}^{\mu}\Phi_j^{(i)}\,dx = \left(\frac{\Gamma(k+\mu)}{\Gamma(k)}\right)^2\frac{2}{2k-1}\delta_{kj}, \tag{50}$$

where $\mathcal{D}^{\mu}$ can be either ${}^{RL}_{-1}\mathcal{D}_x^{\mu}$ or ${}^{C}_{-1}\mathcal{D}_x^{\mu}$, when $i=1$, and can be either ${}^{RL}_{x}\mathcal{D}_1^{\mu}$ or ${}^{C}_{x}\mathcal{D}_1^{\mu}$ when $i=2$.

- **First derivatives**:

$$\frac{d\Phi_n^{(1)}(x)}{dx} = \mu(1+x)^{\mu-1}P_{n-1}^{-\mu,\mu}(x) + \frac{n}{2}(1+x)^{\mu}P_{n-2}^{1-\mu,1+\mu}(x), \tag{51}$$

$$\frac{d\Phi_n^{(2)}(x)}{dx} = -\mu(1-x)^{\mu-1}P_{n-1}^{\mu,-\mu}(x) + \frac{n}{2}(1-x)^{\mu}P_{n-2}^{1+\mu,1-\mu}(x). \tag{52}$$

- **Special values**:

$$\Phi_n^{(1)}(-1) = 0, \tag{53}$$

$$\Phi_n^{(1)}(+1) = 2^{\mu}\binom{n-1-\mu}{n-1}, \tag{54}$$

$$\Phi_n^{(2)}(+1) = 0, \tag{55}$$

$$\Phi_n^{(2)}(-1) = (-1)^{n-1}\Phi_n^{(1)}(+1). \tag{56}$$

4. Part II: Singular fractional Sturm–Liouville problems of kind I & II

In the second part of the paper, we begin with our definition of the singular fractional Sturm–Liouville of first kind I (SFSLP-I) and second kind II (SFSLP-II) of order $\nu = 2\mu \in (0,2)$, with parameters $-1<\alpha<2-\mu$, and $-1<\beta<\mu-1$ in SFSLP-I ($i=1$), and $-1<\alpha<\mu-1$, and $-1<\beta<2-\mu$ in SFSLP-II ($i=2$), for $x\in[-1,1]$ as

$$^{RL}\mathcal{D}^{\mu}\left\{(1-x)^{\alpha+1}(1+x)^{\beta+1}\,{}^{C}\mathcal{D}^{\mu}\mathcal{P}^{(i)}(x)\right\} + \Lambda^{(i)}(1-x)^{\alpha+1-\mu}(1+x)^{\beta+1-\mu}\mathcal{P}^{(i)}(x) = 0, \tag{57}$$

where the fractional order $\mu\in(0,1)$ and $i\in\{1,2\}$, where $i=1$ denotes the SFSLP-I in which ${}^{RL}\mathcal{D}^{\mu} \equiv {}^{RL}_{x}\mathcal{D}_{+1}^{\mu}$ and ${}^{C}\mathcal{D}^{\mu} \equiv {}^{C}_{-1}\mathcal{D}_x^{\mu}$; also $i=2$ corresponds to the RFSLP-II where ${}^{RL}\mathcal{D}^{\mu} \equiv {}^{RL}_{-1}\mathcal{D}_x^{\mu}$ and ${}^{C}\mathcal{D}^{\mu} \equiv {}^{C}_{x}\mathcal{D}_{+1}^{\mu}$. The singular boundary-value problem is subject to the following boundary conditions

$$\mathcal{P}^{(i)}\left((-1)^i\right) = 0, \tag{58}$$

$$\left\{{}^{RL}\mathcal{I}^{1-\mu}\left[p(x)\,{}^{C}\mathcal{D}^{\mu}\mathcal{P}^{(i)}(x)\right]\right\}_{x=(-1)^{i+1}} = 0, \tag{59}$$

where ${}^{RL}\mathcal{I}^{1-\mu} \equiv {}^{RL}_{x}\mathcal{I}_{+1}^{1-\mu}$ when $i=1$ in SFSLP-I, and ${}^{RL}\mathcal{I}^{1-\mu} \equiv {}^{RL}_{-1}\mathcal{I}_x^{1-\mu}$ in case of $i=2$ in SFSLP-II; $p(x) = (1-x)^{\alpha+1}\times(1+x)^{\beta+1}$, used in the fractional differential operator in (57), vanishes at the boundary ends $x=\pm1$. We also note that the weight function $w(x) = (1-x)^{\alpha+1-\mu}(1+x)^{\beta+1-\mu}$ in (57) is a non-negative function.

Theorem 4.1. *The eigenvalues of SFSLP-I & -II (57)–(59) are real-valued, moreover, the eigenfunctions corresponding to distinct eigenvalues of SFSLP-I & -II (57)–(59) are orthogonal with respect to the weight function*

$$w(x) = (1-x)^{\alpha+1-\mu}(1+x)^{\beta+1-\mu}.$$

Proof. *Part a*: Let $\mathcal{L}_i^{\alpha,\beta;\mu}$ be the fractional differential operator of order 2μ as

$$\mathcal{L}_i^{\alpha,\beta;\mu} := {}^{RL}\mathcal{D}^{\mu}\left\{(1-x)^{\alpha+1}(1+x)^{\beta+1}\,{}^{C}\mathcal{D}^{\mu}(\cdot)\right\}, \tag{60}$$

and assume that $\Lambda^{(i)}$ is the eigenvalue of (57)–(59) corresponding the eigenfunction $\eta^{(i)}(x)$, where $i \in \{1,2\}$. Then the following set of equations are valid for $\eta^{(i)}(x)$

$$\mathcal{L}_i^{\alpha,\beta;\mu}\eta^{(i)}(x) + \Lambda^{(i)}w(x)\eta^{(i)}(x) = 0 \tag{61}$$

subject to the boundary conditions

$$\eta^{(i)}\big((-1)^i\big) = 0,$$
$$\left\{{}^{RL}\mathcal{I}^{1-\mu}\left[p(x)\,{}^{C}\mathcal{D}^{\mu}\eta^{(i)}(x)\right]\right\}_{x=(-1)^{(i+1)}} = 0,$$

and its complex conjugate $\bar{\eta}^{(i)}(x)$

$$\mathcal{L}_I^{\alpha,\beta;\mu}\bar{\eta}^{(i)}(x) + \bar{\Lambda}^{(i)}w(x)\bar{\eta}^{(i)}(x) = 0, \tag{62}$$

corresponding to the following boundary conditions

$$\bar{\eta}^{(i)}\big((-1)^i\big) = 0,$$
$$\left\{{}^{RL}\mathcal{I}^{1-\mu}\left[p(x)\,{}^{C}\mathcal{D}^{\mu}\bar{\eta}^{(i)}(x)\right]\right\}_{x=(-1)^{i+1}} = 0.$$

Now, we multiply (61) by $\bar{\eta}^{(i)}(x)$, and (62) by $\eta^{(i)}(x)$ and subtract them as

$$\big(\Lambda^{(i)} - \bar{\Lambda}^{(i)}\big)w(x)\eta^{(i)}(x)\bar{\eta}^{(i)}(x) = \eta^{(i)}(x)\mathcal{L}_i^{\alpha,\beta;\mu}\bar{\eta}^{(i)}(x) - \bar{\eta}^{(i)}(x)\mathcal{L}_i^{\alpha,\beta;\mu}\eta^{(i)}(x). \tag{63}$$

Integrating over the interval $[-1,1]$ and utilizing the fractional integration-by-parts (9) and (10), we obtain

$$\begin{aligned}\big(\Lambda^{(i)} - \bar{\Lambda}^{(i)}\big)\int_{-1}^{+1} w(x)\big|\eta^{(i)}(x)\big|^2\,dx = {} & -\eta\big((-1)^{i+1}\big)\left\{{}^{RL}\mathcal{I}^{1-\mu}\left[p(x)\,{}^{C}\mathcal{D}^{\mu}\eta^{(i)}(x)\right]\right\}_{x=(-1)^{i+1}} \\ & + \bar{\eta}^{(i)}\big((-1)^{i+1}\big)\left\{{}^{RL}\mathcal{I}^{1-\mu}\left[p(x)\,{}^{C}\mathcal{D}^{\mu}\bar{\eta}^{(i)}(x)\right]\right\}_{x=(-1)^{i+1}} \\ & + \eta\big((-1)^i\big) - \bar{\eta}\big((-1)^i\big),\end{aligned} \tag{64}$$

where we re-iterate that ${}^{RL}\mathcal{I}^{1-\mu} \equiv {}^{RL}_{x}\mathcal{I}^{1-\mu}_{+1}$ and ${}^{C}\mathcal{D}^{\mu} \equiv {}^{C}_{-1}\mathcal{D}^{\mu}_{x}$ when $i=1$ in SFSLP-I, also ${}^{RL}\mathcal{I}^{1-\mu} \equiv {}^{RL}_{-1}\mathcal{I}^{1-\mu}_{x}$ and ${}^{C}\mathcal{D}^{\mu} \equiv {}^{C}_{x}\mathcal{D}^{\mu}_{+1}$in case of $i=2$, i.e., SFSLP-II. Now, by applying the boundary conditions for $\eta^{(i)}(x)$ and $\bar{\eta}^{(i)}(x)$ we obtain

$$\big(\Lambda^{(i)} - \bar{\Lambda}^{(i)}\big)\int_{-1}^{+1} w(x)\big|\eta^{(i)}(x)\big|^2\,dx = 0. \tag{65}$$

Therefore, $\Lambda^{(i)} = \bar{\Lambda}^{(i)}$ because $\eta^{(i)}(x)$ is a non-trivial solution of the problem, and $w(x)$ is non-negative in interval $[-1,1]$.

Part b: Now, we prove the second statement on the orthogonality of the eigenfunctions with respect to the weight function $w(x)$. Assume that $\eta_1^{(i)}(x)$ and $\eta_2^{(i)}(x)$ are two eigenfunctions corresponding to two distinct eigenvalues $\Lambda_1^{(i)}$ and $\Lambda_2^{(i)}$, respectively. Then they both satisfy (57)–(59) as

$$\mathcal{L}_i^{\alpha,\beta;\mu}\eta_1^{(i)}(x) + \Lambda_1^{(i)}w(x)\eta_1^{(i)}(x) = 0 \tag{66}$$

subject to

$$\eta_1^{(i)}\big((-1)^i\big) = 0,$$
$$\left\{{}^{RL}\mathcal{I}^{1-\mu}\left[p(x)\,{}^{C}\mathcal{D}^{\mu}\eta_1^{(i)}(x)\right]\right\}_{x=(-1)^{i+1}} = 0,$$

and

$$\mathcal{L}_i^{\alpha,\beta;\mu}\eta_2^{(i)}(x) + \Lambda_2^{(i)}w(x)\eta_2^{(i)}(x) = 0, \tag{67}$$

corresponding to the following boundary conditions

$$\eta_2\big((-1)^i\big) = 0,$$
$$\left\{{}^{RL}\mathcal{I}^{1-\mu}\left[p(x)\,{}^{C}\mathcal{D}^{\mu}\eta_2^{(i)}(x)\right]\right\}_{x=(-1)^{(i+1)}} = 0.$$

It can be shown that

$$\big(\Lambda_1^{(i)} - \Lambda_2^{(i)}\big) w(x)\eta_1^{(i)}(x)\eta_2^{(i)}(x) = \eta_1^{(i)}(x)\mathcal{L}_i^{\alpha,\beta;\mu}\eta_2^{(i)}(x) - \eta_2^{(i)}(x)\mathcal{L}_i^{\alpha,\beta;\mu}\eta_1^{(i)}(x). \tag{68}$$

Integrating over the interval $[-1, 1]$ yields

$$\begin{aligned}\big(\Lambda_1^{(i)} - \Lambda_2^{(i)}\big)\int_{-1}^{+1} w(x)\eta^{(i)}(x)\eta_2^{(i)}(x)\,dx = -\eta_1(+1)\big\{{}^{RL}\mathcal{I}^{1-\mu}\big[p(x)\,{}^{C}\mathcal{D}^{\mu}\eta_1^{(i)}(x)\big]\big\}_{x=+1}\\ + \eta_2^{(i)}(+1)\big\{{}^{RL}\mathcal{I}^{1-\mu}\big[p(x)\,{}^{C}\mathcal{D}^{\mu}\eta_2^{(i)}(x)\big]\big\}_{x=+1}\\ + \eta_1^{(i)}(-1) - \eta_2^{(i)}(-1),\end{aligned} \tag{69}$$

and using fractional integration-by-parts (9) and (10), also since $\Lambda_1^{(i)} - \Lambda_2^{(i)} \neq 0$, we obtain

$$\int_{-1}^{+1} w(x)\eta_1^{(i)}(x)\eta_2^{(i)}(x)\,dx = 0, \tag{70}$$

which completes the proof. □

Theorem 4.2. *The exact eigenfunctions of SFSLP-I (57)–(59), when $i = 1$, are given as*

$$\mathcal{P}_n^{(1)}(x) = {}^{(1)}\mathcal{P}_n^{\alpha,\beta,\mu}(x) = (1+x)^{-\beta+\mu-1} P_{n-1}^{\alpha-\mu+1,-\beta+\mu-1}(x), \tag{71}$$

and the corresponding distinct eigenvalues are

$$\Lambda_n^{(1)} = {}^{(1)}\Lambda_n{}^{\alpha,\beta,\mu} = -\frac{\Gamma(n-\beta+\mu-1)\Gamma(n+\alpha+1)}{\Gamma(n-\beta-1)\Gamma(n+\alpha-\mu+1)}, \tag{72}$$

and furthermore, the exact eigenfunctions to SFSLP-II (57)–(59), in case of $i = 2$, are given as

$$\mathcal{P}_n^{(2)}(x) = {}^{(2)}\mathcal{P}_n^{\alpha,\beta,\mu}(x) = (1-x)^{-\alpha+\mu-1} P_{n-1}^{-\alpha+\mu-1,\beta-\mu+1}(x), \tag{73}$$

and the corresponding distinct eigenvalues are

$$\Lambda_n^{(2)} = {}^{(2)}\Lambda_n^{\alpha,\beta,\mu} = -\frac{\Gamma(n-\alpha+2\mu-1)\Gamma(n+\beta+1)}{\Gamma(n-\alpha+\mu-1)\Gamma(n+\beta-\mu+1)}. \tag{74}$$

Proof. The proof follows similar steps as shown in Theorem 3.4. Hence, we only prove (71) and (72) in detail.

From (71), it is clear that ${}^{(1)}\mathcal{P}_n^{\alpha,\beta,\mu}(-1) = 0$. So, we need to make sure that the other boundary condition is satisfied. Since ${}^{(1)}\mathcal{P}_n^{\alpha,\beta,\mu}(-1) = 0$, the property (7) helps in replacing ${}_{-1}^{C}\mathcal{D}_x^{\mu}$ by ${}_{-1}^{RL}\mathcal{D}_x^{\mu}$. Consequently,

$$\begin{aligned}&\big\{{}^{RL}_{x}\mathcal{I}_{+1}^{1-\mu}\big[p(x)\,{}_{-1}^{C}\mathcal{D}_x^{\mu}\,{}^{(1)}\mathcal{P}_{n-1}^{\alpha,\beta,\mu}(x)\big]\big\}_{x=+1}\\ &\quad= \big\{{}^{RL}_{x}\mathcal{I}_{+1}^{1-\mu}\big[p(x)\,{}_{-1}^{RL}\mathcal{D}_x^{\mu}\,{}^{(1)}\mathcal{P}_{n-1}^{\alpha,\beta,\mu}(x)\big]\big\}_{x=+1}\\ &\quad= \big\{{}^{RL}_{x}\mathcal{I}_{+1}^{1-\mu}\big[p(x)\,{}_{-1}^{RL}\mathcal{D}_x^{\mu}\big((1+x)^{-\beta+\mu-1} P_{n-1}^{\alpha-\mu+1,\,-\beta+\mu-1}(x)\big)\big]\big\}_{x=+1}\end{aligned}$$

and by carrying out the fractional RL derivative using Lemma 3.2

$$\begin{aligned}&= \left\{{}^{RL}_{x}\mathcal{I}_{+1}^{1-\mu}\left[p(x)\frac{\Gamma(n-1-\beta+\mu)}{\Gamma(n-1-\beta)}(1+x)^{-1-\beta} P_{n-1}^{1+\alpha,-1-\beta}\right]\right\}_{x=+1}\\ &= \frac{\Gamma(n-1-\beta+\mu)}{\Gamma(n-1-\beta)}\big\{{}^{RL}_{x}\mathcal{I}_{+1}^{1-\mu}\big[(1-x)^{1+\alpha} P_{n-1}^{1+\alpha,-1-\beta}\big]\big\}_{x=+1},\end{aligned}$$

and by working out the fractional integration using Lemma 3.3 we obtain

$$\frac{\Gamma(n-1-\beta+\mu)}{\Gamma(n-1-\beta)}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-\mu+1)}\big\{(1-x)^{2+\alpha-\mu} P_{n-1}^{1+\alpha,-1-\beta}\big\}_{x=+1} = 0.$$

The next step is to show that (71) satisfies (57) with eigenvalues (72). First, we take a fractional integration of order μ on both sides of (57) and substitute (71). Then, again by replacing the Caputo derivative by the Riemann–Liouville one, thanks to (7), we obtain

$$(1-x)^{\alpha+1}(1+x)^{\beta+1}\,{}^{RL}_{-1}\mathcal{D}^{\mu}_{x}\big[(1+x)^{-\beta+\mu-1}P^{\alpha-\mu+1,-\beta+\mu-1}_{n-1}(x)\big]$$
$$=-\Lambda^{(1)}\,{}^{RL}_{x}\mathcal{I}^{\mu}_{+1}\big\{(1-x)^{\alpha+1-\mu}P^{\alpha-\mu+1,-\beta+\mu-1}_{n-1}(x)\big\}.$$

Finally, the fractional derivative on the left-hand side and the fractional integration on the right-hand side is worked out using (22) and (24) as

$$\frac{\Gamma(n-1-\beta+\mu)}{\Gamma(n-1-\beta)}(1-x)^{\alpha+1}P^{\alpha+1,-\beta-1}_{n-1}(x)=-\Lambda^{(1)}\frac{\Gamma(n+\alpha+\mu+1)}{\Gamma(n+\alpha+1)}(1-x)^{\alpha+1}P^{\alpha+1,-\beta-1}_{n-1}(x).$$

By a similar argument on the $(1-x)^{\alpha+1}P^{\alpha+1,-\beta-1}_{n-1}(x)$ being non-zero almost everywhere, we can cancel this term out on both sides and obtain

$$\Lambda^{(1)}\equiv{}^{(1)}\Lambda^{\alpha,\beta,\mu}_{n}=-\frac{\Gamma(n-\beta+\mu-1)\Gamma(n+\alpha+1)}{\Gamma(n-\beta-1)\Gamma(n+\alpha-\mu+1)}.$$

Now, we need to check Theorem 4.1, to see if (72) verifies that the eigenvalues are indeed real-valued and distinct, and the orthogonality of the eigenfunctions with respect to $w(x)=(1-x)^{1+\alpha-\mu}(1+x)^{1+\beta-\mu}$ is valid:

$$\int_{-1}^{1}w(x)\,{}^{(1)}\mathcal{P}^{\alpha,\beta,\mu}_{k}(x)\,{}^{(1)}\mathcal{P}^{\alpha,\beta,\mu}_{j}(x)\,dx=\int_{-1}^{1}w(x)\big[(1+x)^{-\beta+\mu-1}\big]^{2}P^{\alpha-\mu+1,-\beta+\mu-1}_{k-1}(x)P^{\alpha-\mu+1,-\beta+\mu-1}_{j-1}(x)\,dx$$
$$=\int_{-1}^{1}(1-x)^{1+\alpha-\mu}(1+x)^{-\beta+\mu-1}P^{\alpha-\mu+1,-\beta+\mu-1}_{k-1}(x)P^{\alpha-\mu+1,-\beta+\mu-1}_{j-1}(x)\,dx$$
$$=\int_{-1}^{1}(1-x)^{\alpha^{*}}(1+x)^{\beta^{*}}P^{\alpha^{*},\beta^{*}}_{k-1}(x)P^{\alpha^{*},\beta^{*}}_{j-1}(x)\,dx=C^{\alpha^{*},\beta^{*}}(k-1)\delta_{kj},$$

where $\alpha^{*}=\alpha-\mu+1$, $\beta^{*}=-\beta+\mu-1$, and denoted by $C^{\alpha^{*},\beta^{*}}(k)$ is the orthogonality constant of the family of Jacobi polynomials.

The simplicity of the eigenvalues can be also shown in a similar fashion as Part c in the proof of Theorem 3.4, and this completes the proof. □

Lemma 4.3. *The shifted eigenfunctions to SFSLP-I & -II, denoted by* ${}^{(i)}\widetilde{\mathcal{P}}^{\alpha,\beta,\mu}_{n}(t)$, *are given as*

$$^{(i)}\widetilde{\mathcal{P}}^{\alpha,\beta,\mu}_{n}(t)=2^{\widetilde{\mu}^{(i)}}\sum_{j=0}^{n-1}(-1)^{n+j-1}\binom{n-1+j}{j}\binom{n+(-1)^{i+1}\widetilde{\mu}^{(1)}-1}{n-1-j}t^{j+\widetilde{\mu}^{(i)}},\tag{75}$$

where in case of the SFLSP-I $(i=1)$, $t\in[0,1]$, $\widetilde{\mu}^{(1)}=-\beta+\mu-1$ *and* $0<\widetilde{\mu}^{(1)}<\mu$, *and for SFSLP-II* $(i=2)$, $t\in[-1,0]$, $\widetilde{\mu}^{(2)}=-\alpha+\mu-1$ *also* $0<\widetilde{\mu}^{(2)}<\mu$.

Proof. The proof follows the one in Lemma 3.5. □

Theorem 4.4. *The shifted eigensolutions to (57),* ${}^{(i)}\widetilde{\mathcal{P}}^{\alpha,\beta,\mu}_{n}(t)$, *form a complete hierarchical basis for the finite-dimensional space of poly-fractonomials* $\mathbb{F}_{n-1+\widetilde{\mu}^{(i)}}$, *where* $\widetilde{\mu}^{(1)}=-\beta+\mu-1$ *and* $\widetilde{\mu}^{(2)}=-\alpha+\mu-1$, *where* $0<\widetilde{\mu}^{(1)}<\mu$, *also* $0<\widetilde{\mu}^{(2)}<\mu$.

Proof. The proof follows the one in Theorem 3.11. □

The growth of the magnitude in the eigenvalues of SFSLP-I, $|\Lambda^{(1)}_{n}|$, exhibits a similar behavior as one observed in RFSLP-I & -II. However, there are another two degrees of freedom in the choice of parameters α and β, which affect the magnitude of the eigenvalues. It turns out that in case of SFSLP-I $(i=1)$, the optimal highest magnitude is achieved when $\alpha\to 2-\mu$ and $\beta\to -1$, $\forall\mu\in(0,1)$. The growth of the $|\Lambda^{(1)}_{n}|$ corresponding to three values of $\mu=0.35$, $\mu=0.5$, and $\mu=0.99$ is shown in Fig. 4. Again, we observe about the two growth modes of $|\Lambda^{(1)}_{n}|$, depending on either $\mu\in(0,1/2)$, where a sublinear growth in $|\Lambda^{1}_{n}|$ is observed, or, $\mu\in(1/2,1)$, where a superlinear–subquadratic growth mode is valid; the case $\mu=1/2$ leads to an exactly linear growth mode. Corresponding to the aforementioned fractional-orders μ, in Fig. 5, we plot the eigenfunctions of SFSLP-I, $\mathcal{P}^{(1)}_{n}(x)$, of different orders and corresponding to different values of μ used in Fig. 4. In a similar fashion, we compare the eigensolutions with the corresponding standard Jacobi polynomials $P^{\alpha-\mu+1,-\beta+\mu-1}_{n}(x)$ in each plot.

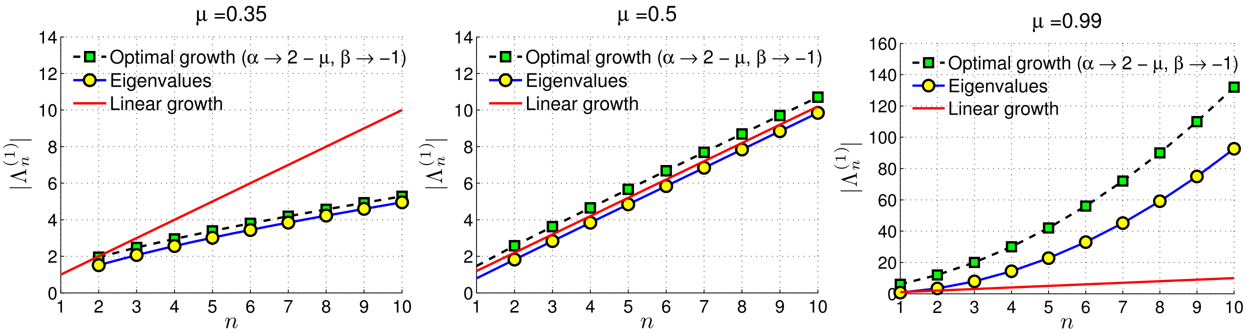

Fig. 4. Magnitude of the eigenvalues of SFSLP-I, $|\Lambda_n^{(1)}|$, versus n, corresponding to $\alpha = 0$ and $\beta = -0.7$, corresponding to different fractional-order $\mu = 0.35$, left: sublinear growth, $\mu = 0.5$; middle: linear growth, and $\mu = 0.99$; right: superlinear–subquadratic growth. Here we compare the growth of the eigenvalues to the optimal case when $\alpha \to 2 - \mu$ and $\beta \to -1$.

In Fig. 6, the growth of the magnitude in $\Lambda_n^{(2)}$, corresponding to three values of $\mu = 0.35$, $\mu = 0.5$, and $\mu = 0.99$ is plotted. In SFSLP-II ($i = 2$), the optimal highest magnitude in the eigenvalues is achieved when $\alpha \to -1$ and $\beta \to 2 - \mu$. Moreover, in Fig. 7, we plot the eigenfunctions of SFSLP-II, $\mathcal{P}_n^{(2)}(x)$, of different fractional-orders and corresponding to different μ used in Fig. 6. This time, we compare the eigensolutions with the corresponding standard Jacobi polynomials $P_n^{-\alpha+\mu-1,\beta-\mu+1}(x)$ in each plot.

4.1. Properties of the eigen-solutions to SFSLP-I & -II

We list a number of properties of the eigensolutions to SFSLP-I & -II as follows.

- **Non-polynomial nature**:
  From (71) and (73), the eigenfunctions exhibit a non-polynomial (fractal) behavior, thanks to the fractonomial multipliers $(1+x)^{-\beta+\mu-1}$ in SFSLP-I and $(1-x)^{-\alpha+\mu-1}$ in SFSLP-II. Indeed, these poly-fractonomials are the generalization of those introduced in RFSLP (27) and (29). We realize that when $\alpha \to -1$ and $\beta \to -1$ simultaneously, the eigen-solutions to the singular problems SFSLP-I & -II, only *asymptotically*, approach to that of the regular counterparts. However, special attention should be taken due to the fact that when $\alpha \to -1$ and $\beta \to -1$, the governing equations (57) then become *non-singular* and equivalent to the regular problems RFSLP-I & -II (16) at the first place.
  Here, we refer to ${}^{(i)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)$ as the *generalization of the whole family of the Jacobi poly-fractonomial* corresponding to the triple ${}^{\alpha,\beta,\mu}$, where $-1 < \alpha < 2-\mu$, and $-1 < \beta < \mu - 1$ in SFSLP-I ($i = 1$), and $-1 < \alpha < \mu - 1$, and $-1 < \beta < 2 - \mu$ in SFSLP-II ($i = 2$).

- **Asymptotic eigenvalues** $\Lambda_n^{(i)}$:
  The growth in the magnitude of eigenvalues in SFSLP with n is dependent on three parameters: the fractional derivative order μ, α and β. From (72) and (74), it is easy to show that α and β only affect the magnitude and not the behavior (i.e., order) of the growth. As shown in (30), since $\mu \in (0,1)$, there are two modes of growth in the magnitude of $\Lambda_n^{(i)}$ referred to as *sublinear* mode corresponding to $0 < \mu < 1/2$, and *superlinear–subquadratic* mode which corresponds to $1/2 < \mu < 1$. Particularly, when $\mu = 1/2$, the eigenvalues grow linearly with n. The optimal highest magnitude of $\Lambda_n^{(1)}$ achieved when $\alpha \to 2 - \mu$ and $\beta \to -1$ in SFSLP-I, and in case of the SFSLP-II when $\alpha \to -1$ and $\beta \to 2 - \mu$ the optimal eigenvalues are obtained. The asymptotic cases are summarized as

$$\left|\Lambda_n^{(i)}\right| = \begin{cases} n^2, & \mu \to 1, \\ n, & \mu \to 1/2, \\ 1, & \mu \to 0. \end{cases} \tag{76}$$

- **Recurrence relations**:
  A recurrence relations is obtained for the Jacobi poly-fractonomials ${}^{(i)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)$ as

$$\begin{aligned}
&{}^{(1)}\mathcal{P}_1^{\alpha,\beta,\mu}(x) = (1+x)^{-\beta+\mu-1},\\
&{}^{(1)}\mathcal{P}_2^{\alpha,\beta,\mu}(x) = \frac{1}{2}(1+x)^{-\beta+\mu-1}\left[\alpha+\beta-2\mu+2+(\alpha-\beta+2)x\right],\\
&\vdots\\
&a_n\, {}^{(1)}\mathcal{P}_{n+1}^{\alpha,\beta,\mu}(x) = (b_n + c_n x)\, {}^{(1)}\mathcal{P}_n^{\alpha,\beta,\mu}(x) - d_n\, {}^{(1)}\mathcal{P}_{n-1}^{\alpha,\beta,\mu}(x),\\
&a_n = 2n(n+\alpha-\beta)(2n+\alpha-\beta-2),
\end{aligned}$$

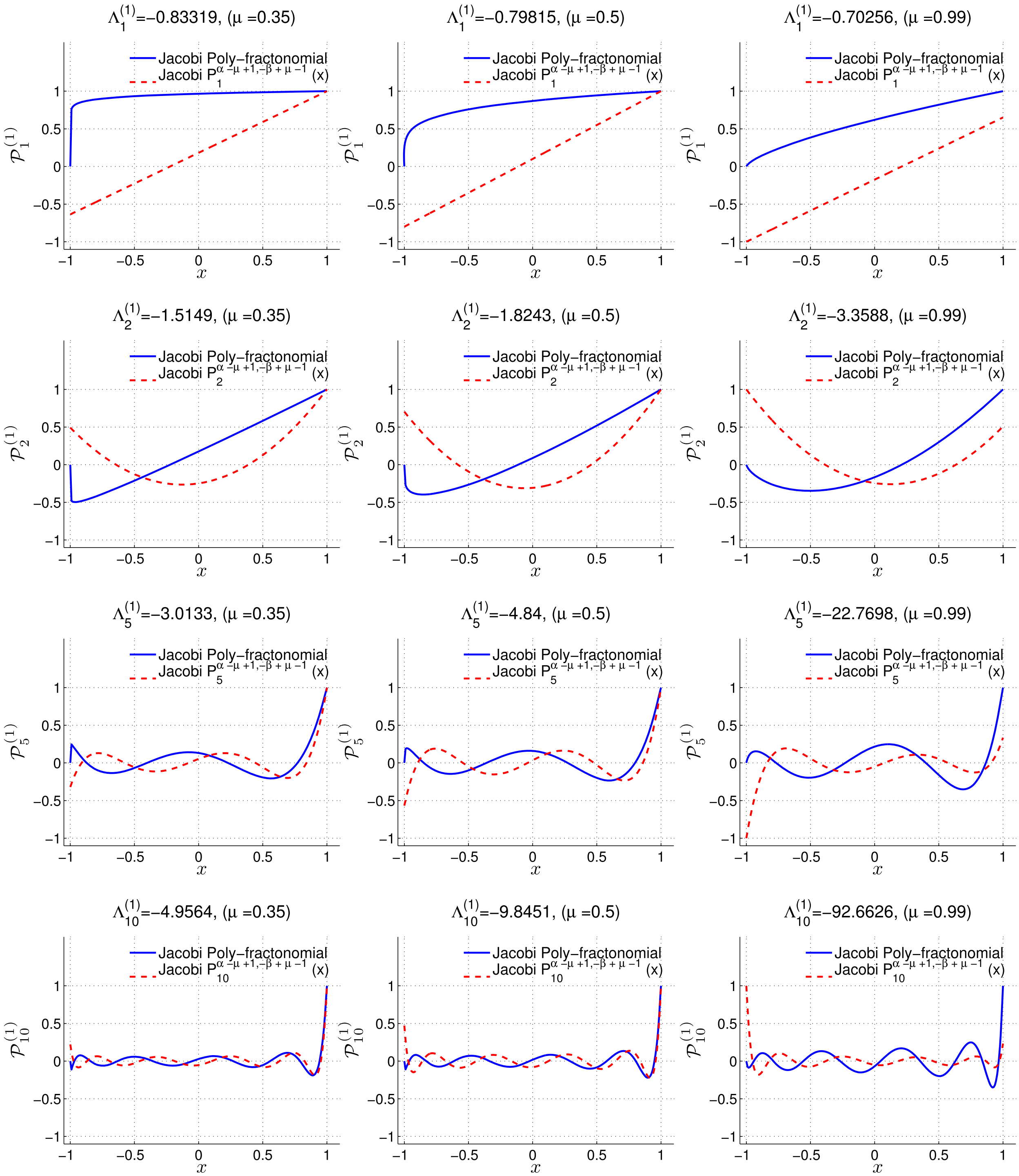

Fig. 5. Eigenfunctions of SFSLP-I, $\mathcal{P}_n^{(1)}$, versus x, for $n = 1$ (first row), $n = 2$ (second row), $n = 5$ (third row), and $n = 10$ (last row), corresponding to the fractional-order $\mu = \nu/2 = 0.35$ (left column), $\mu = \nu/2 = 0.5$ (middle column), and $\mu = \nu/2 = 0.99$ (right column). Here, we take the same values $\alpha = 0$ and $\beta = -0.7$, as shown in Fig. 4.

$$
\begin{aligned}
b_n &= (2n - \alpha + \beta - 1)(\alpha - \beta)(\alpha + \beta - 2\mu + 2), \\
c_n &= (2n - \alpha + \beta)(2n - \alpha + \beta - 1)(2n - \alpha + \beta - 2), \\
d_n &= 2(n - \alpha + \mu - 2)(n + \beta - \mu)(2n - \alpha + \beta),
\end{aligned} \tag{77}
$$

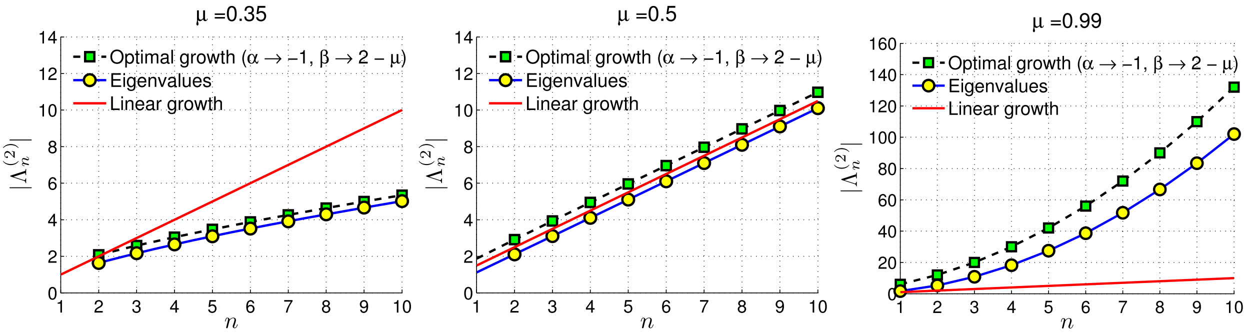

Fig. 6. Magnitude of the eigenvalues of SFSLP-II, $|\Lambda_n^{(2)}|$, versus n, corresponding to $\alpha=-0.7$ and $\beta=0$, corresponding to different fractional-order $\mu=0.35$, left: sublinear growth, $\mu=0.5$; middle: linear growth, and $\mu=0.99$; right: superlinear–subquadratic growth. Here we compare the growth of the eigenvalues to the optimal case when $\alpha\to-1$ and $\beta\to 2-\mu$.

and

$$
\begin{aligned}
&{}^{(2)}\mathcal{P}_1^{\alpha,\beta,\mu}(x)=(1-x)^{-\alpha+\mu-1},\\
&{}^{(2)}\mathcal{P}_2^{\alpha,\beta,\mu}(x)=\frac{1}{2}(1-x)^{-\alpha+\mu-1}\big[-\alpha-\beta+2\mu-2+(-\alpha+\beta+2)x\big],\\
&\quad\vdots\\
&a_n^{*}\,{}^{(2)}\mathcal{P}_{n+1}^{\alpha,\beta,\mu}(x)=\big(b_n^{*}+c_n^{*}x\big)\,{}^{(2)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)-d_n^{*}\,{}^{(2)}\mathcal{P}_{n-1}^{\alpha,\beta,\mu}(x),\\
&a_n^{*}=2n(n-\alpha+\beta)(2n-\alpha+\beta-2),\\
&b_n^{*}=(2n-\alpha+\beta-1)(\alpha-\beta)(\alpha+\beta-2\mu+2),\\
&c_n^{*}=(2n+\alpha+\beta)(2n+\alpha+\beta-1)(2n+\alpha+\beta-2),\\
&d_n^{*}=2(n+\alpha-\mu)(n-\beta+\mu-2)(2n+\alpha-\beta).
\end{aligned}
\tag{78}
$$

- **Orthogonality**:

$$
\int_{-1}^{1}(1-x)^{\alpha+1-\mu}(1+x)^{\beta+1-\mu}\,{}^{(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x)\,{}^{(i)}\mathcal{P}_j^{\alpha,\beta,\mu}(x)\,dx={}^{(i)}\mathcal{C}_k^{\alpha,\beta}\delta_{kj},
\tag{79}
$$

where

$$
{}^{(1)}\mathcal{C}_k^{\alpha,\beta}=\frac{2^{\alpha-\beta+1}}{2k+\alpha-\beta-1}\,\frac{\Gamma(k+\alpha-\mu+1)\Gamma(k-\beta+\mu-1)}{(k-1)!\Gamma(k+\alpha-\beta)},
$$

and

$$
{}^{(2)}\mathcal{C}_k^{\alpha,\beta}=\frac{2^{-\alpha+\beta+1}}{2k-\alpha+\beta-1}\,\frac{\Gamma(k-\alpha+\mu-1)\Gamma(k+\beta-\mu+1)}{(k-1)!\Gamma(k-\alpha+\beta)}.
$$

- **Fractional derivatives**:

$$
{}^{RL}_{-1}\mathcal{D}_x^{-\beta+\mu+1}\big({}^{(1)}\mathcal{P}_n^{\alpha,\beta,\mu}\big)={}^{C}_{-1}\mathcal{D}_x^{-\beta+\mu+1}\big({}^{(1)}\mathcal{P}_n^{\alpha,\beta,\mu}\big)=\frac{\Gamma(n+\mu)}{\Gamma(n)}P_{n-1}^{\alpha-\beta,0}(x),
\tag{80}
$$

and

$$
{}^{RL}_{x}\mathcal{D}_1^{-\alpha+\mu-1}\big({}^{(2)}\mathcal{P}_n^{\alpha,\beta,\mu}\big)={}^{C}_{x}\mathcal{D}_1^{-\alpha+\mu-1}\big({}^{(2)}\mathcal{P}_n^{\alpha,\beta,\mu}\big)=\frac{\Gamma(n+\mu)}{\Gamma(n)}P_{n-1}^{0,\beta-\alpha}(x),
\tag{81}
$$

where $P_{n-1}^{\alpha-\beta,0}(x)$ and $P_{n-1}^{0,\beta-\alpha}(x)$ denote the standard Jacobi polynomials.

- **First derivatives**:

$$
\begin{aligned}
\frac{d}{dx}\big({}^{(1)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)\big)=&(-\beta+\mu-1)(1+x)^{-\beta+\mu-2}P_{n-1}^{\alpha-\mu+1,-\beta+\mu-1}(x)\\
&+\frac{1}{2}(n+\alpha-\beta)(1+x)^{-\beta+\mu-1}P_{n-2}^{\alpha-\mu+2,-\beta+\mu}(x),
\end{aligned}
$$

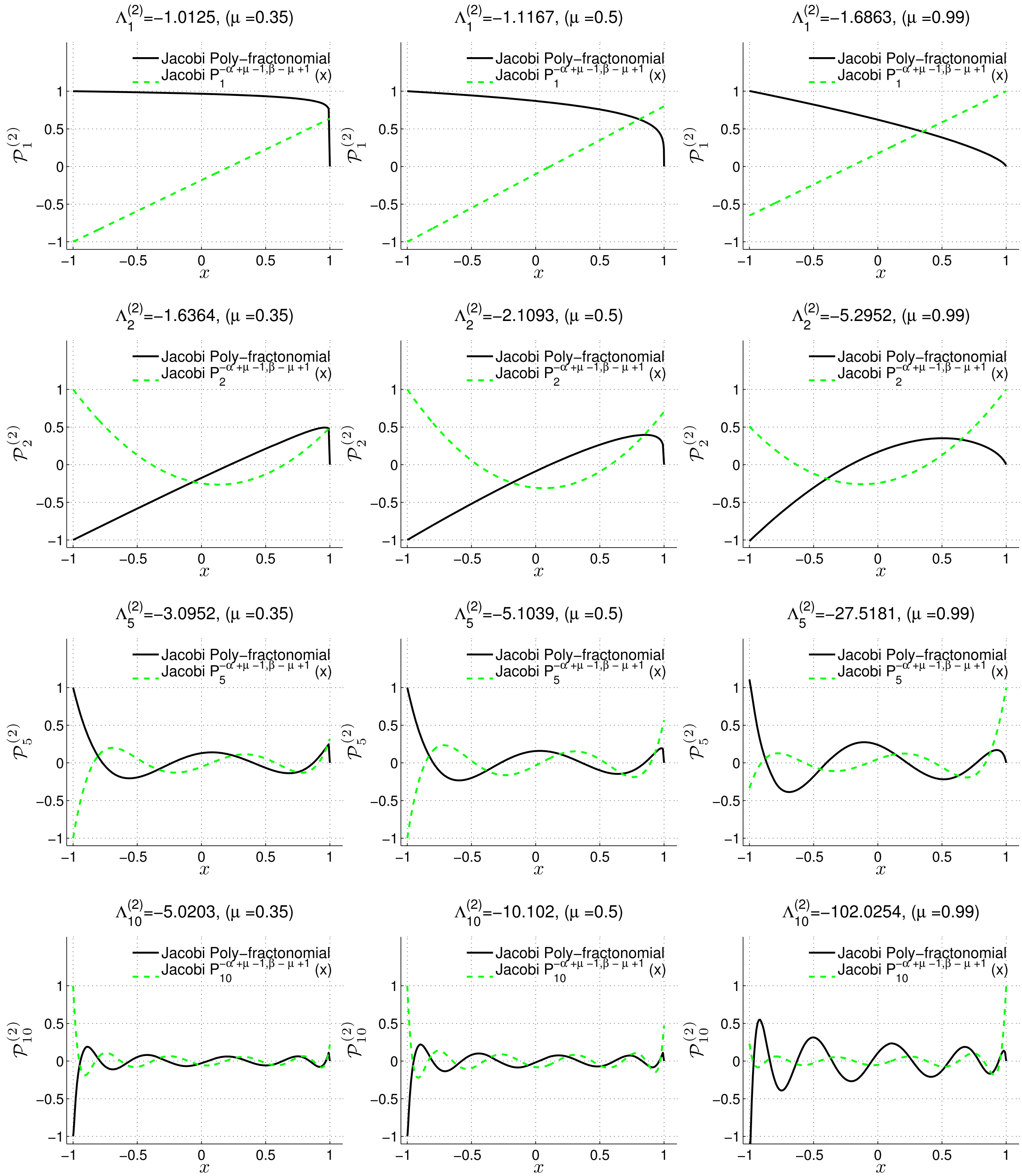

Fig. 7. Eigenfunctions of SFSLP-II, $\mathcal{P}_n^{(2)}$, versus x, for $n = 1$ (first row), $n = 2$ (second row), $n = 5$ (third row), and $n = 10$ (last row), corresponding to the fractional-order $\mu = \nu/2 = 0.35$ (left column), $\mu = \nu/2 = 0.5$ (middle column), and $\mu = \nu/2 = 0.99$ (right column). Here, we take the same values $\alpha = -0.7$ and $\beta = 0$, as shown in Fig. 6.

and

$$\frac{d}{dx}\left({}^{(2)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)\right) = (+\alpha - \mu + 1)(1-x)^{-\alpha+\mu-2} P_{n-1}^{-\alpha+\mu-1,\beta-\mu+1}(x) + \frac{1}{2}(n - \alpha + \beta)(1-x)^{-\alpha+\mu-1} P_{n-2}^{-\alpha+\mu,\beta-\mu+2}(x).$$

- **Special values**:

$$
\begin{aligned}
&{}^{(1)}\mathcal{P}_n^{\alpha,\beta,\mu}(-1)=0,\\
&{}^{(1)}\mathcal{P}_n^{\alpha,\beta,\mu}(+1)=2^{-\beta+\mu-1}\binom{n+\alpha-\mu}{n-1},
\end{aligned}
\tag{82}
$$

and

$$
\begin{aligned}
&{}^{(2)}\mathcal{P}_n^{\alpha,\beta,\mu}(+1)=0,\\
&{}^{(2)}\mathcal{P}_n^{\alpha,\beta,\mu}(-1)=2^{-\alpha+\mu-1}\binom{n+\beta-\mu}{n-1}.
\end{aligned}
\tag{83}
$$

5. Numerical approximation

As discussed in Section 4.1, taking $\alpha=\beta=-1$ in SFLP-I & -II essentially eliminates the singularity in the definition of SFSLP-I & -II (57). Accordingly, we are not allowed to take such values for α and β, unless asymptotically, in the SFSLP-I & -II. However, the Jacobi poly-fractonomials ${}^{(i)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)$, $i\in\{1,2\}$, regardless of where they are coming from, are the generalization of the poly-fractonomials $\Phi_n^{(i)}(x)$ which are known as the eigenfunctions of FSLP-I & -II. Therefore, we can represent the whole family of the Jacobi poly-fractonomials ${}^{(i)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)$ as

$$
{}^{(i)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)=\begin{cases}\text{eigenfunctions of RFSLPs in (16)}, & \alpha=\beta=-1,\\ \text{eigenfunctions of SFSLPs in (57)}, & \text{otherwise},\end{cases}
\tag{84}
$$

where $i\in\{1,2\}$.

By Theorems 3.11 and 4.4, we can employ such basis functions for numerical approximation. In such setting, we can study the approximation properties of the family of Jacobi poly-fractonomials ${}^{(i)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)$ in a unified fashion. To this end, we represent a function $f(x)\in L^2_w[-1,1]$ as

$$
f(x)\approx f_N(x)=\sum_{n=1}^{N}\hat f_n\,{}^{(i)}\mathcal{P}_n^{\alpha,\beta,\mu}(x),\quad x\in[-1,1]
\tag{85}
$$

where $f(x)$ satisfied the same boundary conditions as ${}^{(i)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)$ in (85). Now, the main question is how fast the expansion coefficients $\hat f_n$ decay. By multiplying (85) by $\mathcal{L}_i^{\alpha,\beta;\mu}({}^{(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x))$, $k=1,2,\dots,N$, and integrating in the interval $[-1,1]$, we obtain

$$
\int_{-1}^{1} f(x)\mathcal{L}_i^{\alpha,\beta;\mu}\big({}^{(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x)\big)\,dx=\int_{-1}^{1}\left(\sum_{n=1}^{N}\hat f_n\,{}^{(i)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)\right)\mathcal{L}_i^{\alpha,\beta;\mu}\big({}^{(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x)\big)\,dx,
$$

where $\mathcal{L}_i^{\alpha,\beta;\mu}({}^{(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x))$ on the right-hand side can be substituted by the right-hand side of (57), i.e., $-\Lambda_n^{(i)}w(x)\,{}^{(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x)$ as

$$
\int_{-1}^{1} f(x)\mathcal{L}_i^{\alpha,\beta;\mu}\big({}^{(i)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)\big)\,dx=\sum_{n=1}^{N}-\hat f_n\,\Lambda_n^{(i)}\int_{-1}^{1}(1-x)^{\alpha+1-\mu}(1+x)^{\beta+1-\mu}\,{}^{(i)}\mathcal{P}_n^{\alpha,\beta,\mu}(x)\,{}^{(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x),
$$

and thanks to the orthogonality property (79) we get

$$
\hat f_k=\frac{-1}{{}^{(i)}\mathcal{C}_k^{\alpha,\beta}\,\Lambda_k^{(i)}}\int_{-1}^{1} f(x)\mathcal{L}_i^{\alpha,\beta;\mu}\big({}^{(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x)\big)\,dx,
$$

or equivalently by (60),

$$
\hat f_k=\frac{-1}{{}^{(i)}\mathcal{C}_k^{\alpha,\beta}\,\Lambda_k^{(i)}}\int_{-1}^{1} f(x)\,{}^{RL}\mathcal{D}^{\mu}\big\{(1-x)^{\alpha+1}(1+x)^{\beta+1}\,{}^{C}\mathcal{D}^{\mu}\big({}^{(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x)\big)\big\}\,dx.
\tag{86}
$$

We recall that $i=1$ corresponds to ${}^{RL}\mathcal{D}^{\mu} \equiv {}^{RL}_{x}\mathcal{D}^{\mu}_{+1}$ and ${}^{C}\mathcal{D}^{\mu} \equiv {}^{C}_{-1}\mathcal{D}^{\mu}_{x}$, also when $i=2$ we have ${}^{RL}\mathcal{D}^{\mu} \equiv {}^{RL}_{-1}\mathcal{D}^{\mu}_{x}$ and ${}^{C}\mathcal{D}^{\mu} \equiv {}^{C}_{x}\mathcal{D}^{\mu}_{+1}$. Now, by carrying out the fractional integration-by-parts (9) and (10), we get

$$\hat{f}_k = \frac{-1}{{}^{(i)}C_k^{\alpha,\beta}\Lambda_k^{(i)}}\int_{-1}^{1}(1-x)^{\alpha+1}(1+x)^{\beta+1}\left({}^{C}\mathcal{D}^{\mu} f(x)\right)\left({}^{C}\mathcal{D}^{\mu\,(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x)\right)dx, \tag{87}$$

which is equivalent to

$$\hat{f}_k = \frac{-1}{{}^{(i)}C_k^{\alpha,\beta}\Lambda_k^{(i)}}\int_{-1}^{1}(1-x)^{\alpha+1}(1+x)^{\beta+1}\left({}^{C}\mathcal{D}^{\mu\,(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x)\right)\left({}^{C}\mathcal{D}^{\mu} f(x)\right)dx - {}^{(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x)\,{}^{RL}_{x}\mathcal{I}^{\mu}_{x_R} f(x)\Big|_{x=-1}^{+1}. \tag{88}$$

We realize that the last term in (88) is identically zero. Again, by the fractional integration-by-parts (9) and (10), we obtain

$$\hat{f}_k = \frac{-1}{{}^{(i)}C_k^{\alpha,\beta}\Lambda_k^{(i)}}\int_{-1}^{1}{}^{(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x)\,{}^{RL}\mathcal{D}^{\mu}\left\{(1-x)^{\alpha+1}(1+x)^{\beta+1}\,{}^{C}\mathcal{D}^{\mu} f(x)\right\}dx,$$

or equivalently

$$\hat{f}_k = \frac{-1}{{}^{(i)}C_k^{\alpha,\beta}\Lambda_k^{(i)}}\int_{-1}^{1}{}^{(i)}\mathcal{P}_k^{\alpha,\beta,\mu}(x)\,\mathcal{L}_i^{\alpha,\beta;\mu}\left[f(x)\right]dx,$$

if denoted by $f_{(1)}(x) \equiv \mathcal{L}_i^{\alpha,\beta;\mu}[f(x)] \in L^2_w[-1,1]$. By carrying out the fractional integration-by-parts another $(m-1)$ times, and setting $f_{(m)}(x) \equiv \mathcal{L}_i^{\alpha,\beta;\mu}[f_{(m-1)}(x)] \in L^2_w[-1,1]$, we obtain

$$|\hat{f}_k| \approx \frac{C}{|\Lambda_k^{(i)}|^m}\left\| f_{(m)}(x)\right\|_{L^2_w}, \quad k=1,2,\ldots,N. \tag{89}$$

Consequently, if the function $f(x) \in C^{\infty}[-1,1]$, we recover the exponential decay of the expansion coefficients $\hat{f}_k$.

Remark 5.1. Although when $0<\mu<1/2$ the magnitude of the eigenvalues grows sublinearly, such decay behavior does not affect fundamentally the exponential character of the decay in the coefficients if $f(x)$ possesses the required regularity.

5.1. Numerical tests

In the following examples, we test the convergence rate in approximating some poly-fractonomials in addition to some other type of functions involving fractional character. By Theorems 3.11 and 4.4, we can exactly represent any poly-fractonomial $F_{N+\mu}$ of order $N+\mu$ in terms of the first N regular Jacobi fractal basis functions (16), or alternatively, using the first N singular Jacobi fractal basis functions (75). However, this is not the case when other types of basis functions, such as the standard (shifted) Legendre polynomials $\widetilde{P}_n(x)$, are employed.

We first approximate the simplest fractal function $f(t)=\sqrt{t}$ using our regular and singular Jacobi poly-fractonomials, where we see that only *one* term is needed to exactly represent the fractonomial, i.e., $f(t)=f_1(t)$. To make a comparison, we also plot the L^2-norm error in terms of N, the number of expansion terms in (85) in Fig. 8 (left), when the standard Legendre polynomials are employed as the basis functions. Moreover, we represent the poly-fractonomial $f(t)=t^{1/3}+t^{4+1/3}+t^{7+1/3}$ by our regular and singular Jacobi poly-fractonomials to compare the efficiency of such expansion functions to other standard polynomial bases. The fast (super) spectral convergence of the our fractal basis functions shown in Fig. 8 (right), compared to that of the Legendre expansion, highlights the efficiency of Jacobi poly-fractonomial basis functions in approximating non-polynomial functions. Next, we approximate another two functions which are not poly-fractonomials. In Fig. 9, we show the L^2-norm error in (85), where the convergence to $f(t)=t^{1/3}\sin(2t)$ is shown on the left and the error in the approximation of $f(t)=\sin(3\sqrt{t})$ is plotted on the right. Once again, we observe spectral (exponential) convergence of (85) when the regular and the singular eigenfunctions are employed as the basis functions, compared to the case when the standard Legendre polynomials are employed. Finally, we also test how well smooth functions are approximated using a non-polynomial basis in Fig. 10. As expected, we see that the Legendre polynomial basis is outperforming the poly-fractonomial basis but only slightly and we still observe exponential convergence of the latter. Here we employed $\mu=1/2$ for the both RFSLP and SFSLP bases but other choices are also possible to optimize the convergence rate.

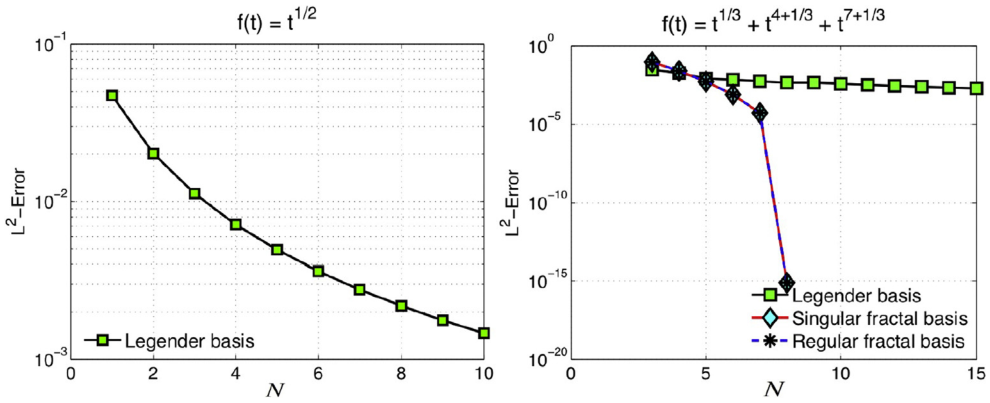

Fig. 8. L^2-norm error $\|f(t)-f_N(t)\|_{L^2}$ versus N, the number of expansion terms in (85) when Legendre polynomials are used as the basis functions. Here, $f(t)$ is a poly-fractonomial; left: $f(t)=\sqrt{t}$, where only one term, i.e., ${}^{(i)}\mathcal{P}_1^{\alpha,\beta,\mu}$ is needed to exactly capture $\sqrt{t}$, and right: $f(t)=t^{1/3}+t^{4+1/3}+t^{7+1/3}$; here $\alpha=\beta=0$.

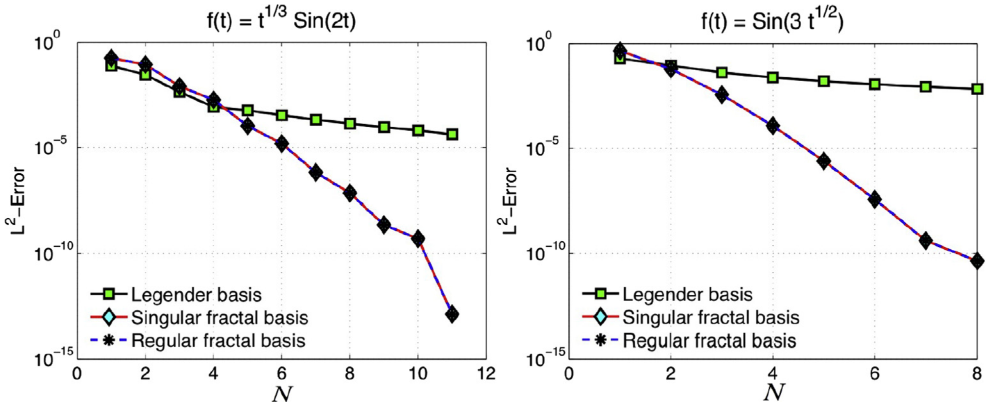

Fig. 9. L^2-norm error $\|f(t)-f_N(t)\|_{L^2}$ versus N, the number of expansion terms in (85), where $f(t)$ is not a poly-fractonomial; left: $f(t)=t^{1/3}\sin(2t)$, and right: $f(t)=\sin(3\sqrt{t})$; here $\alpha=\beta=0$.

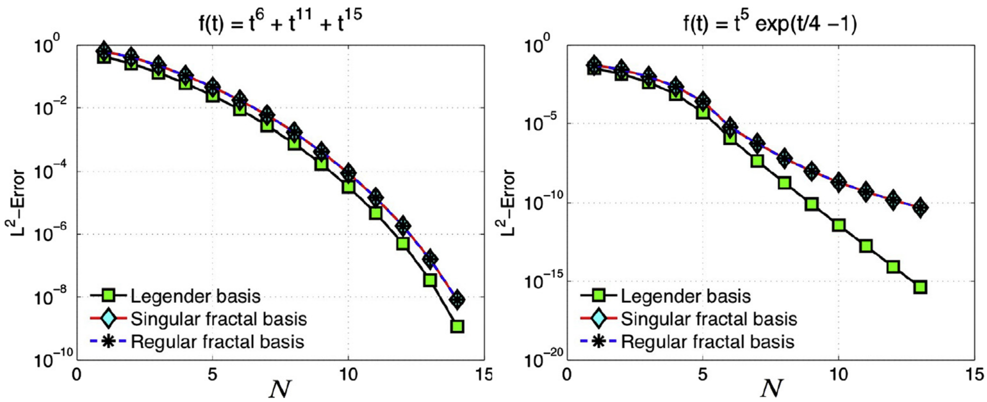

Fig. 10. L^2-norm error $\|f(t)-f_N(t)\|_{L^2}$ versus N, the number of expansion terms in (85), where $f(t)$ is a polynomial; left: $f(t)=t^6+t^{11}+t^{15}$, and right: $f(t)=t^5\exp t/4-1$; here $\alpha=\beta=0$.

6. Summary

We have considered a regular fractional Sturm–Liouville problem of two kinds RFSLP-I and RFSLP-II of order $\nu\in(0,2)$, [15] with the fractional differential operators both of Riemann–Liouville and Caputo type, of the same fractional-order $\mu=\nu/2\in(0,1)$. This choice, in turn, motivated a proper fractional integration-by-parts. In the first part of the paper, we obtained the analytical eigensolutions to RFSLP-I & -II as non-polynomial functions, which we defined as Jacobi *poly-fractonomials*. These eigenfunctions were shown to be orthogonal with respect to the weight function, associated with the

RFSLP-I & -II. In addition, these eigenfunctions were shown to be hierarchical, and a useful recursive relation was obtained for each type of the eigenfunctions. Moreover, a detailed list of other important properties of such poly-fractonomials was presented at the end of the first part of the paper.

We extended the fractional operators to a new family of singular fractional Sturm–Liouville problems of two kinds, SFSLP-I and SFSLP-II, in the second part of the paper. We showed that the regular boundary-value problems RFSLP-I & -II are indeed asymptotic cases for the singular counterparts SFSLP-I & -II. We also proved that the eigenvalues of the singular problems are real-valued and the eigenfunctions corresponding to distinct eigenvalues are orthogonal. Subsequently, we obtained the eigen-solutions to SFSLP-I & -II analytically, also as non-polynomial functions, which completed the whole family of the Jacobi poly-fractonomials. In a similar fashion, a number of useful properties of such eigensolutions was introduced.

Finally, we analyzed the numerical approximation properties of the eigensolutions to RFSLP-I & -II and SFSLP-I & -II in a unified fashion. The exponential convergence in approximating fractal functions such as poly-fractonomials in addition to some other fractal functions such as fractional trigonometric functions highlighted the efficiency of the new fractal basis functions compared to Legendre polynomials.

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