APMA2821F **Computational Linear Algebra** **Homework # 3**

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1. **Convection terms in two dimensions**. A convection term can be added to the two-dimensional model problem in the form

$$-\epsilon(u_{xx} + u_{yy}) + au_x = f(x).$$

Using a 2D equidistant grid and second-order central finite difference approximations, find the system of linear equations associated with this problem. What condition must be met by a and ϵ for the associated matrix to be diagonally dominant?

2. **Gauss-Seidel eigenvalues and eigenvectors**.

   (a) Show that the eigenvalue problem for the Gauss-Seidel interaction matrix, $R_G\mathbf{w} = \lambda\mathbf{w}$, may be expressed in the form $U\mathbf{w} = (D - L)\lambda I\mathbf{w}$, where $A = D - L - U$.

   (b) Write out the equations of this system and note the boundary condition $w_0 = w_n = 0$. Look for solutions of this system of equations of the form $w_j = \mu^j$, where $\mu \in \mathbf{C}$ must be determined. Show that the boundary conditions can be satisfied only if $\lambda = \lambda_k = \cos^2\left(\frac{k\pi}{n}\right)$, $1 \le k \le n - 1$.

   (c) Show that the eigenvector associated with λ_k is $w_{k,j} = \lambda_k^{j/2} \sin\left(\frac{jk\pi}{n}\right)$.

3. **Effect of full weighting**. Verify that $I_h^{2h}\mathbf{w}_k^h = \cos^2\left(\frac{k\pi}{2n}\right)\mathbf{w}_k^{2h}$, where $w_{k,j}^h = \sin\left(\frac{jk\pi}{n}\right)$, $1 \le k < \frac{n}{2}$, and I_h^{2h} is the full weighting operator.

4. **Effect of full weighting**. Verify that $I_h^{2h}\mathbf{w}_{k'}^h = -\sin^2\left(\frac{k\pi}{2n}\right)\mathbf{w}_k^{2h}$, where $k' = n-k$, $1 \le k < \frac{n}{2}$, and I_h^{2h} is the full weighting operator. What happens to $\mathbf{w}_{n/2}^h$ under full weighting?

5. **Complementary modes**. Show that the complementary modes $\{\mathbf{w}_k^h, \mathbf{w}_{k'}^h\}$ on Ω^h are related by $w_{k',j}^h = (-1)^{j+1}w_{k,j}^h$, where $k' = n - k$.