Homework 5

Problem 12.1

Construct an example to show that the convergence of the Jacobi method does not necessarily imply that the Gauss-Seidel method will converge.

Solution:
For example, let
\[ A = \begin{pmatrix} 3 & -5 & 2 \\ 5 & 4 & 3 \\ 2 & 5 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \]

Then we need 204 iterations to get the solution with Jacob method, but Gauss-Seidel method doesn’t converge. Further, we can calculate \( \rho(B_J) = 0.9129 < 1 \), and \( \rho(B_{GS}) = 2.1630 > 1 \), so Jacobi method converges with any initial guess, while it’s not true for GS method.

Problem 12.13

Apply the Jacobi, gauss-seidel, and SOR(with optimal relaxation factor) methods to the system in Example 12.10 and verify the statement made there about the number of iterations for different methods.

\[ A = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \]

Solutions:
The eigenvalues of \( B_J \) are \( 0.1036, 0.2500, -0.1036, -0.2500, 0.6036, -0.6036. \) \( \rho(B_J) = 0.6036, \rho(B_{GS}) = 0.3643, \omega_{opt} = \frac{2}{1+\sqrt{1-(0.6036)^2}} = 1.1128. \) It took five iterations for the SOR method with \( \omega_{opt} \) to converge to the exact solution(up to four significant figures), starting with \( x_{SOR}^{(0)} = (0, 0, ..., 0)^T. \) With the same starting vector \( x^{(0)} \), the Gauss-Seidel method required nine iterations. Also eighteen iterations is requred by the Jacobi method.

```bash
>> B = 4*diag(ones(3,1)) - diag(ones(2,1),1) - diag(ones(2,1),-1); 
>> I = diag(ones(3,1)); 
>> A = [B, -I; -I,B]; 
>> b = zeros(6,1); 
>> b(1)=1;  
>> x0=zeros(6,1); 
```
Problem M12.1

Run the programs `jacobi`, `gaused`, and `sucov` (choosing $\omega$ as an optional relaxation parameter) from MATCOM on the $500 \times 500$ matrix $A$ of the same type as that of Example 12.10 with the same starting vector $x^{(0)} = (0, 0, ..., 0)^T$. Find how many iterations each method will take to converge.

Solution:
The $\omega$ for SOR is 1.20374444000930. Here I use the $\epsilon_p = 0.00000001$. With the same starting vector $x^{(0)} = (0, 0, ..., 0)^T$, it took 53 iterations for the Jacobi method, 26 for the GS method, and 11 for SOR method.
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```matlab
>> v = eig(Bj);
>> norm(v,inf)
ans =
   0.749960836036112
>> w=2/(1+sqrt(1-ans*ans))
w =
   1.20374444000930

>> [x,iter] = jacobi(A,x0,b,ep,numitr);

>> [x,iter] = gaused(A,x0,b,ep,numitr);

>> [x0,iter] = sucov(A,x0,b,w,ep,numitr);
```