1. Partial derivatives. Recall that in class we discussed the notion of a partial derivative. In particular, for a smooth function $u(x, t)$ defined on some open rectangle $(a, b) \times (c, d)$ in two dimensional space, the partial derivative with respect to $x$ at a point $(x, t)$ in this rectangle is defined to be

$$u_x(x, t) = \frac{\partial}{\partial x} u(x, t) = \lim_{\delta \to 0} \frac{u(x + \delta, t) - u(x, t)}{\delta},$$

and the partial derivative with respect to $t$ (denoted $u_t(x, t)$) is defined similarly. One can compute the partial with respect to $x$ by pretending that $t$ is a constant and computing the ordinary derivative with respect to $x$, and likewise for the partial with respect to $t$. Compute the partials, both with respect to $x$ and $t$, for the indicated functions on the indicated rectangles.

- $\sin xt$, $(x, t) \in (-\infty, \infty) \times (-\infty, \infty)$
- $\sin(x^2 + t^2)$, $(x, t) \in (-\infty, \infty) \times (-\infty, \infty)$
- $\log(x/t)$, $(x, t) \in (0, \infty) \times (0, \infty)$
- $\cos x \cos t$, $(x, t) \in (-\infty, \infty) \times (-\infty, \infty)$.

2. Suppose that $L > 0$ is a constant. Show that for any value of $\lambda$ the function

$$f(x) = \sin \lambda x$$

is a solution to the ordinary differential equation

$$f_{xx}(x) + \lambda^2 f(x) = 0.$$

Next find that particular constants $\lambda > 0$ such that $f$ not only satisfies this differential equation, but also the boundary conditions

$$f(0) = 0, f(L) = 0.$$

3. Recall that two $n$-dimensional vectors $\mathbf{y}$ and $\mathbf{z}$ are said to be orthogonal (or perpendicular) if the dot product is zero:

$$\mathbf{y} \cdot \mathbf{z} = \sum_{i=1}^{n} y_i z_i = 0.$$
We use a similar definition if we want to speak of two continuous functions defined on the interval $[0, L]$ as being orthogonal. The inner product of two continuous functions $f$ and $g$ mapping $[0, L]$ to the real numbers is defined to be

$$\langle f, g \rangle = \frac{1}{L} \int_0^L f(x)g(x)dx.$$ 

Show that the following functions are all mutually orthogonal on $[0, L]$:

$$\sin \frac{n\pi x}{L}, n = 1, 2, \ldots.$$ 

4. Show also that this inner product possesses the familiar linearity of the ordinary dot product. In other words, if $f$, $g$ and $h$ are continuous functions mapping $[0, L]$ to the reals and if $a$ and $b$ are real numbers, then

$$\langle af + bg, h \rangle = a \langle f, h \rangle + b \langle g, h \rangle.$$