Applied Math 9

Problem Set 3 for Zero Sum Games

1. For each of the following systems of equations, use the determinant to decide if there is a unique solution.

   \[(2, 1) \cdot (x_1, x_2) = 3, (4, 2) \cdot (x_1, x_2) = 3\]
   \[(1, 4) \cdot (x_1, x_2) = 2, (-1, 4) \cdot (x_1, x_2) = 3\]
   \[(2, 1, 0) \cdot (x_1, x_2, x_3) = 1, (1, 1, 1) \cdot (x_1, x_2, x_3) = 0\]
   \[(1, 2, 1) \cdot (x_1, x_2, x_3) = 0, (3, 2, 1) \cdot (x_1, x_2, x_3) = 0\]

2. A set of points \(S\) in \(n\)-dimensional space is called convex if given any two points \((x_1, x_2, \ldots, x_n)\) and \((y_1, y_2, \ldots, y_n)\) in \(S\), the line joining these points also lies in \(S\). In other words, for any \(0 \leq t \leq 1\),

   \[(x_1, x_2, \ldots, x_n)t + (y_1, y_2, \ldots, y_n)(1 - t) \in S\]

   Are half-spaces convex? Is the intersection of 2 or more half spaces convex?

3. In class we discussed how to turn the Holmes-Moriarty lower game with mixed strategies into a maximization problem. Show that the upper can also be turned into a maximization problem.

4. A function \(f\) on \(n\)-dimensional space is called affine if it is the sum of a dot product and a constant. In other words, for some known vector \((a_1, a_2, \ldots, a_n)\) and number \(b\) and with variable \((x_1, x_2, \ldots, x_n)\),

   \[f(x_1, x_2, \ldots, x_n) = (a_1, a_2, \ldots, a_n) \cdot (x_1, x_2, \ldots, x_n) + b.\]

   Suppose that \(S\) is convex. A point \((x_1, x_2, \ldots, x_n) \in S\) is called a non-extreme point if there is a direction \((v_1, v_2, \ldots, v_n)\), such that small movements back and forth in the direction \((v_1, v_2, \ldots, v_n)\) around \((x_1, x_2, \ldots, x_n)\) are still in \(S\). (In precise mathematical terms, we would say that there is \(c > 0\) such that

   \[(x_1, x_2, \ldots, x_n) + t(v_1, v_2, \ldots, v_n) \in S\]

   as long as \(|t| < c\).) All other points are called extreme points. Do you think an affine function can take on its maximum value at a non-extreme point and be strictly smaller at all other points in \(S\)?
5. Let $A$ be an $n \times n$ matrix with determinant 0 and let $B$ be an other $n \times n$ matrix. What are $\det(AB)$ and $\det(BA)$? (It may be useful to think of $A$ and $B$ as defining transformations and use the volume interpretation of determinant.)