Applied Math 9

Problem set 1 for Zero Sum Games

Notation. Given an $n \times m$ matrix $A$ with entries $a_{ij}$, the transpose of $A$, denoted by $A^T$ (and sometimes by $A'$) is the $m \times n$ matrix whose entry in the $i$th row and $j$th column is $a_{ji}$. E.g., if $A$ is

$$
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix},
$$

then $A^T$ is

$$
\begin{pmatrix}
1 & 4 \\
2 & 5 \\
3 & 6
\end{pmatrix}.
$$

1. We use $\mathbb{R}^n$ to denote $n$-dimensional Euclidean space (which we identify with the set of all $n$-dimensional real vectors). If $x = [x_1, x_2, x_3]^T$ and

$$
A = \begin{pmatrix}
4 & 2 & 9 \\
3 & 5 & 4 \\
1 & 2 & 3 \\
0 & 1 & 2
\end{pmatrix},
$$

then the mapping from $x$ to $Ax$ is a mapping from $\mathbb{R}^3$ to $\mathbb{R}^4$ (often denoted $\mathbb{R}^3 \to \mathbb{R}^4$). What is the value of the map when $x = [0, 1, 0]^T$?

2. Let

$$
A = \begin{pmatrix}
3 \\
5
\end{pmatrix}, B = \begin{pmatrix}
1 & 1 \\
1 & 2
\end{pmatrix}, C = \begin{pmatrix}
1 \\
2
\end{pmatrix}.
$$

What is $ABC$? Is it equal to $CBA$? Does the order in which you do the computations matter (i.e., is $A(BC) = (AB)C$)?

3. Write the following system of linear equations in the matrix form $Au = b$, where $u$ is the column vector $[x, y, z]^T$.

$$
\begin{align*}
2x - 3y + 4z &= -19 \\
6x + 4y - 2z &= 8 \\
x + 5y + 4z &= 23.
\end{align*}
$$
• Solve the system using matlab.

4. • Let $A$ be an arbitrary $3 \times 3$ matrix and let

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

Show that $IA = A$ and also that $AI = A$. We call $I$ the multiplicative identity matrix for all $3 \times 3$ matrices. The analogous formula defines the identity for $n \times n$ matrices.

• Use $A = rand(3)$ to generate a “random” matrix and $I = eye(3)$ in matlab and then test your conclusion.

5. An $n \times n$ matrix $A$ is called symmetric if $A = A^T$.

• Show that for any $3 \times 3$ symmetric matrix $A$ and any 3-dimensional row vector $x$,

$$(xA)^T = Ax^T. $$

Hint: write $A$ as

$$ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}, $$

and $x = [x_1, x_2, x_3]$.

6. Consider a game with an $n \times m$ payoff matrix $A$. Suppose we identify action $i$ of Player 1 with the $n$-dimensional row vector $y^{n,i} = x$, where

$$x_k = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i. \end{cases} $$

Likewise we identify action $j$ of Player 2 with the $m$-dimensional column vector $z^{m,j} = x$, where

$$x_k = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases} $$

Express the payoff when Player 1 chooses action $i$ and Player 2 chooses action $j$ in terms of the matrices $y^{n,i}, z^{m,j}$ and $A$. Hint: the outcome is an $1 \times 1$ matrix, i.e., a scalar. Check the dimensions.
7. Consider the payoff matrix

\[
\begin{pmatrix}
3 & -3 & -2 & -4 \\
-4 & -2 & -1 & 1 \\
1 & -1 & 2 & 0
\end{pmatrix}.
\]

- Suppose that Player 1 has the advantage, and Player 2 must choose first. What is the best (minimax) choice for Player 2? If the roles are reversed, what is the best (maximin) choice for Player 1?
- Does the game have value? If so, what are the saddle point strategies.