Applied Mathematics 120, Spring of 2005

Derivation of the Dynamic Programming Equation for an Example Done in Class

A detailed derivation of the dynamic programming equation. The key is that we can think of the final position \( x_N \) in two ways. One is as a function of \((u_n, \ldots, u_{N-1}) \) and \( x_n \), and the second is as a function of \((u_{n+1}, \ldots, u_{N-1}) \) and \( x_{n+1} \), which itself depends on \( u_n \) and \( x_n \):

\[
V_n(x_n) = \max_{(u_n, \ldots, u_{N-1})} \left[ \sum_{j=n}^{N-1} B[1-u_j] + g(x_N) \right]
\]

\[
= \max_{(u_n, \ldots, u_{N-1})} \left[ B[1-u_n] + \sum_{j=n+1}^{N-1} B[1-u_j] + g(x_N[x_n, u_n, \ldots, u_{N-1}]) \right]
\]

\[
= \max_{u_n} \left[ B[1-u_n] + \max_{(u_{n+1}, \ldots, u_{N-1})} \left[ \sum_{j=n+1}^{N-1} B[1-u_j] + g(x_N[x_{n+1}[x_n, u_n], u_{n+1}, \ldots, u_{N-1}]) \right] \right]
\]

\[
= \max_{u_n} \left[ B[1-u_n] + V_{n+1}(f(x_n, u_n)) \right].
\]

The first equality is the definition of \( V_n(x_n) \), the second separates out the benefit for the first step after \( n \) and writes \( x_N \) as a function of \((u_n, \ldots, u_{N-1}) \) and \( x_n \), the third writes \( x_N \) as a function of \((u_{n+1}, \ldots, u_{N-1}) \) and \( x_{n+1} \), which itself depends on \( u_n \) and \( x_n \), the fourth uses that the one step benefit does not depend on \( u_{n+1}, \ldots, u_{N-1} \), the fifth uses the formula for how we get \( x_{n+1} \) from \( x_n, u_n \), and the last uses the definition of \( V_{n+1} \).