Let us consider a non-homogeneous linear system of differential equations
\[ \dot{y}(t) = A(t) y + f(t), \]  
(9.1)
where \( A(t) \) is a given \( n \times n \) matrix with continuous coefficients, and \( f(t) \) is known \( n \)-vector column of some functions. Here \( y(t) = (y_1, y_2, \ldots, y_n)^T \) is the vector of \( n \) functions to be determined. Suppose that we know the fundamental matrix for the homogeneous system \( \dot{x}(t) = A(t) x \), which is convenient to write as a collection of vector columns:
\[ X(t) = [x_1(t), x_2(t), \ldots, x_n(t)], \]
where each \( n \)-vector \( x_k(t) \), \( k = 1, 2, \ldots, n \), is a solution of the homogeneous equation:
\[ \dot{x}(t) = A(t) x(t). \]  
(9.2)
It is known that the general solution of the homogeneous equation (9.2) is
\[ y(t) = X(t) c \]
for an arbitrary constant-vector \( c \). The Lagrange’s idea is to replace the constant vector \( c \) with a variable vector \( u(t) \), and seek a particular solution of Eq. (9.1) in the form
\[ y_p(t) = X(t) u(t). \]
By product rule, its derivative is
\[ y'_p(t) = X'(t) u(t) + X(t) u'(t). \]
Hence, upon substituting \( y_p \) into Eq. (9.1), we get
\[ X'(t) u(t) + X(t) u'(t) = A(t) X(t) u(t) + f(t). \]
Since \( X'(t) = A(t) X(t) \), we have
\[ X(t) u'(t) = f(t) \quad \text{or} \quad u'(t) = X^{-1}(t) f(t). \]
Thus, if \( u(t) \) is the solution of the last equation, namely,
\[ u(t) = \int X^{-1}(t) f(t) \, dt + c, \]
where \( c \) is an arbitrary constant vector of integration, then a particular solution becomes

\[
y_p(t) = X(t)u(t) = X(t) \int X^{-1}(t) f(t) \, dt + X(t) c.
\]

We may set \( c \) to be zero since we just need to find a particular solution but not all possible solutions to the nonhomogeneous equation (9.1). When an initial position is specified,

\[
\dot{y}(t) = A(t) y + f(t), \quad y(t_0) = y_0,
\]

it is convenient to rewrite a particular solution of Eq. (9.3) as

\[
y_p(t) = X(t) \int_{t_0}^{t} X^{-1}(s) f(s) \, ds.
\]

Then the solution to the initial value problem (9.3) becomes

\[
y(t) = X(t) \int_{t_0}^{t} X^{-1}(s) f(s) \, ds + X(t) X^{-1}(t_0) y_0.
\]

If \( A \) is a constant square matrix, then the initial value problem

\[
\dot{y} = Ay(t) + f(t), \quad y(t_0) = y_0
\]

has the unique solution:

\[
y(t) = e^{A(t-t_0)} y_0 + \int_{t_0}^{t} e^{A(t-s)} f(s) \, ds = e^{A(t-t_0)} y_0 + e^{At} \int_{t_0}^{t} e^{-As} f(s) \, ds.
\]

**Example 9.1** Find a particular solution of

\[
\dot{y}(t) = A(t) y(t) + f(t), \quad \text{where} \quad A = \frac{1}{t} \begin{bmatrix} 3 & -t \\ 0 & 1 \end{bmatrix}, \quad f(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix},
\]

given that

\[
X(t) = \begin{bmatrix} t^2 & t^3 \\ t & 0 \end{bmatrix}
\]

is a fundamental matrix for the complementary system \( \dot{x} = Ax \).

**Solution.** We seek a particular solution \( y_p(t) \) of the given inhomogeneous vector differential equation in the form

\[
y_p(t) = X(t)u(t),
\]

where \( u(t) \) is unknown vector of functions. Substitution into the equation leads

\[
X(t)u'(t) = f(t) \quad \text{or} \quad u'(t) = X^{-1}(t)f(t)
\]

The inverse of the fundamental matrix is

\[
X^{-1}(t) = \frac{1}{t^3} \begin{bmatrix} 0 & t^2 \\ 1 & -t \end{bmatrix}.
\]
Therefore,

\[ X^{-1}(t)f(t) = \frac{1}{t^3} \begin{bmatrix} 0 & t^2 \\ 1 & -t \end{bmatrix} \begin{bmatrix} t \\ t^2 \end{bmatrix} = \frac{1}{t^2} \begin{bmatrix} t^3 \\ 1 - t^2 \end{bmatrix} = \begin{bmatrix} t \\ t^2 - 1 \end{bmatrix}. \]

Integration yields

\[ u = \int X^{-1}(t)f(t) \, dt = \begin{bmatrix} t^2/2 \\ -t^{-1} - t \end{bmatrix} + c. \]

Multiplication by \( X(t) \) gives a particular solution:

\[ y_p = X(t)u = \begin{bmatrix} -t^4/2 - t^2 \\ t^3/2 \end{bmatrix} + X(t)c. \]

Indeed, differentiating \( y_p \), we get

\[ \dot{y}_p = \begin{bmatrix} -2t^3 - 2t \\ \frac{3}{2} t^2 \end{bmatrix}, \]

which is equal to

\[ Ay_p + f = \frac{1}{t} \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -t^4/2 - t^2 \\ t^3/2 \end{bmatrix} + f = \begin{bmatrix} -2t^3 - 3t \\ t^2/2 \end{bmatrix} + \begin{bmatrix} t \\ t^2 \end{bmatrix} = \begin{bmatrix} -2t^3 - 2t \\ \frac{3}{2} t^2 \end{bmatrix}. \]

**Example 9.2** Consider the vector constant coefficient equation

\[ \dot{y} = Ay + f, \]

where \( A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, f = \begin{bmatrix} t \\ t^2 \end{bmatrix} \). Using Eq. (9.7), we find a particular solution

\[ y_p = e^{At} \int_0^t e^{-A\tau}f(\tau) \, d\tau. \]

Since the exponential matrix function is known to be

\[ e^{At} = e^{2t} \begin{bmatrix} 1 + t & -t \\ t & 1 - t \end{bmatrix}, \quad e^{-At} = e^{-2t} \begin{bmatrix} 1 - t & t \\ -t & 1 + t \end{bmatrix}, \]

we find a particular solution by integration:

\[ y_p = \begin{bmatrix} 1 + t & -t \\ t & 1 - t \end{bmatrix} \left( -\frac{1}{8} \right) \begin{bmatrix} 3 + 6t + 2t^2 + 4t^3 \\ 3 + 6t + 6t^2 + 4t^3 \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} 3 + 6t + 2t^2 \\ 3 + 6t + 6t^2 \end{bmatrix}. \]