Many physical applications lead to systems of ordinary differential equations. Since higher order systems can be converted into equivalent first order case, we do not lose any generality by restricting our attention to the first order case throughout. “Equivalent” means that each solution to the higher order equation uniquely corresponds to a solution to the first order system. Recall that \(|a, b|\) denotes any interval (open, closed, or semi-closed) with end points \(a, b\).

A first order system of ordinary differential equations has the general form

\[
\frac{du_1}{dt} = f_1(t, u_1, \ldots, u_n), \ldots, \frac{du_n}{dt} = f_n(t, u_1, \ldots, u_n).
\]  

(8.1)

The unknowns \(u_1(t), \ldots, u_n(t)\) are scalar functions of a real variable \(t\), which usually represents time. The right-hand side functions \(f_1(t, u_1, \ldots, u_n), \ldots, f_n(t, u_1, \ldots, u_n)\) are given functions of \(n + 1\) variables. It is a custom to denote the derivative \(d/dt\) with respect time variable by a dot, that is, \(\frac{du}{dt} = \dot{u}\). By introducing vector columns \(u = \langle u_1, u_2, \ldots, u_n \rangle^T\) and \(f = \langle f_1, f_2, \ldots, f_n \rangle^T\), we rewrite Eq. (8.1) in a vector form:

\[
\frac{du}{dt} = f(t, u) \quad \text{or} \quad \dot{u} = f(t, u).
\]  

(8.2)

By a solution to the system of differential equations (8.1) on an interval \(|a, b|\) we mean a vector-column function \(u(t)\) with \(n\) components that is defined and continuously differentiable on an interval \([a, b]\), and, moreover, satisfies the given vector equation on its interval of definition. Each solution \(u(t)\) serves to parametrize a curve in an \(n\) dimensional space, also known as a trajectory or orbit of the system.

When we seek a particular solution that starts at the specified point, we impose the initial conditions:

\[
u_1(t_0) = u_{10}, \ u_2(t_0) = u_{20}, \ldots, u_n(t_0) = u_{n0} \quad \text{or} \quad u(t_0) = u_0.
\]  

(8.3)

Here \(t_0\) is a prescribed initial time, while the vector-column \(u_0 = \langle u_{10}, u_{20}, \ldots, u_{n0} \rangle^T\) fixes the initial position of the desired solution. In favorable situations, to be formulated shortly, the initial conditions serve to uniquely specify a solution to the differential system of equations—at least for nearby times. A system of equations (8.2) together with the initial conditions (8.3) form the initial value problem or the Cauchy problem.

A system of differential equations is called \textbf{autonomous} if the right hand side does not explicitly depends upon the time \(t\), and so takes the form

\[
\frac{du}{dt} = f(u) \quad \text{or} \quad \dot{u} = f(u).
\]  

(8.4)
One important class of autonomous first order systems are steady state fluid flows, where \( v = f(u) \) represents the fluid velocity at the position \( u \). The solution \( u(t) \) to the initial value problem (8.4), (8.3) describes the motion of a fluid particle that starts at position \( u_0 \) at time \( t_0 \). The differential equations (8.4) tell us that the fluid velocity at each point on the particle’s trajectory matches the prescribed vector field.

**Theorem:** Let \( f(t, u) \) be a continuous function in a neighborhood of the point \((t_0, u_0)\) in the \((n+1)\)-dimensional space. Then the initial value problem

\[
\dot{u} = f(t, u), \quad u(t_0) = u_0
\]

(8.5)

admits a solution \( u(t) \) that is, at least, defined for nearby times, i.e., when \(|t - t_0| < \delta\) for some positive \( \delta \).

This theorem guarantees that the solution to the initial value problem exists in some neighborhood of the initial position. However, the interval of existence of the solution might be much larger. It is called the **validity interval**, and it is barred by singularities, if any. So the interval of existence can be unbounded, possibly infinite, \(-\infty < t < \infty\). Note that the existence theorem can be readily adapted to any higher order system of ordinary differential equations through converting it into an equivalent first order system by introducing additional variables.

**Theorem:** Let \( f(t, u) \) be a vector-valued function with \( n \) components. If \( f \) is continuous in some domain and satisfies the Lipschitz condition:

\[
\| f(t, u_1) - f(t, u_2) \| \leq \alpha \cdot \| x_1 - x_2 \|,
\]

\( \alpha \) is a positive constant,

then the initial value problem (8.5) has a unique solution on some open interval containing the point \( t_0 \). Here \( \| x \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \) is the norm (length) of the vector \( x = \langle x_1, x_2, \ldots, x_n \rangle^T \).

If \( f(t, u) \) is continuously differentiable, then there exists one and only one solution to the initial value problem (8.5).

As a first consequence, we find that solutions of the autonomous system (8.4) are uniquely determined by its initial data. So the solution trajectories do not vary over time: the functions \( u(t) \) and \( u(t - a) \) parametrize the same curve in the \( n \)-dimensional space. Therefore, all solutions passing through the point \( u_0 \) follow the same trajectory, irrespective of the time they arrive there.

For linear system of differential equations we have more strong result.

**Theorem:** Let the \( n \times n \) matrix-valued function \( A(t) \) and the vector-valued function \( g(t) \) be continuous on the (bounded or unbounded) open interval \([a, b]\) containing the point \( t_0 \). Then the initial value problem

\[
\dot{x} = A(t)x + g(t), \quad x(t_0) = x_0, \quad t_0 \in [a, b],
\]

(8.6)

has a continuous vector-valued solution \( x(t) \) on the interval \([a, b]\).

A system of linear differential equations in normal form (8.6) is called a **vector differential equation**. If the vector column \( f(t) \), usually called **nonhomogeneous term** or **forcing function**, is not identically zero, then Eq. (8.6) is called **nonhomogeneous** or **inhomogeneous**. Otherwise we have a **homogeneous** vector equation

\[
\frac{dx(t)}{dt} = A(t)x(t).
\]

(8.7)
Eq. (8.7) is called the **complementary equation** to nonhomogeneous equation (8.7), and its solution is called the **complementary function**, which contains \( n \) arbitrary constants.

**Theorem: [Superposition Principle for Homogeneous Equations]** Let \( \mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_k(t) \) be a set of solution vectors of the homogeneous system (8.7) on an interval \([a, b]\). Then their linear combination

\[
\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + \cdots + c_k \mathbf{x}_k(t),
\]

where \( c_i, i = 1, 2, \ldots, k \), are arbitrary constants, is also a solution to Eq. (8.7) on the interval.

**Theorem: [Superposition Principle for Inhomogeneous Equations]** Let \( \mathbf{A}(t) \) be an \( n \times n \) matrix function that is continuous on an interval \([a, b]\), and let \( \{ \mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t) \} \) be a fundamental solution set for first order linear system of differential equations

\[
\dot{\mathbf{x}}(t) = \mathbf{A}(t) \mathbf{x}, \quad t \in [a, b].
\]

If \( \mathbf{x}_p(t) \) is any particular solution to the nonhomogeneous vector differential equation

\[
\dot{\mathbf{x}}(t) = \mathbf{A}(t) \mathbf{x} + \mathbf{g}(t), \quad t \in [a, b],
\]

then every solution to Eq. (8.9) is of the form

\[
\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + \cdots + c_n \mathbf{x}_n(t) + \mathbf{x}_p(t) \tag{8.10}
\]

for some constants \( c_1, c_2, \ldots, c_n \). In other words, the general solution of a nonhomogenous linear vector equation (8.9) is the sum of the general solution of the complementary homogeneous equation (8.8) and a particular solution of the inhomogeneous equation (8.9).

We recall the following definition.

A set of \( n \) vector functions \( \mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t) \) is said to be **linearly dependent** if there exists a set of numbers \( c_1, c_2, \ldots, c_n \), with at least one nonzero, such that

\[
c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + \cdots + c_n \mathbf{x}_n(t) \equiv 0 \quad \text{for all } t.
\]

Otherwise these vector-functions are called **linearly independent**.

Two vector functions are linearly dependent if and only if one is a constant multiple of another.

Let \( \mathbf{A}(t) \) be an \( n \times n \) matrix that is continuous on an interval \([a, b]\) \((a < b)\). Any set of \( n \) solutions \( \mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t) \) to the homogeneous vector equation \( \dot{\mathbf{x}}(t) = \mathbf{A}(t) \mathbf{x}(t) \) that is linearly independent on the interval \([a, b]\) is called a **fundamental set of solutions** (or fundamental solution set). Then the \( n \times n \) matrix

\[
\mathbf{X}(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1(t) | \mathbf{x}_2(t) | \cdots | \mathbf{x}_n(t) \end{bmatrix}
\]

having these solution vectors as its column vectors, is called a **fundamental matrix** for the system of differential equations \( \dot{\mathbf{x}}(t) = \mathbf{A}(t) \mathbf{x}(t) \).

**Theorem:** If the \( n \times n \) matrix \( \mathbf{A}(t) \) has continuous entries on some open interval, then the vector differential equation \( \dot{\mathbf{x}} = \mathbf{A}(t) \mathbf{x}(t) \) has an \( n \times n \) fundamental matrix \( \mathbf{X}(t) = [\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)] \). Every solution to this system can be written as

\[
\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + \cdots + c_n \mathbf{x}_n(t) \quad \text{or in matrix form} \quad \mathbf{x}(t) = \mathbf{X}(t) \mathbf{c} \tag{8.11}
\]
for appropriate constants $c_1, c_2, \ldots, c_n$, where \( c = (c_1, c_2, \ldots, c_n)^T \) is a vector column of constants.

Throughout the text, we will refer to (8.11) as the **general solution** to the homogeneous vector differential equation (8.7).

The determinant, $W(t) = \det X(t)$, of a fundamental matrix $X(t) = [x_1(t), x_2(t), \ldots, x_n(t)]$ is called the **Wronskian** of the vector functions $x_1(t), x_2(t), \ldots, x_n(t)$.

**Theorem:** [N. Abel] Let $A(t)$ be an $n \times n$ matrix with entries $a_{ij}(t)$ that are continuous functions on some interval. Let $x_k(t), k = 1, 2, \ldots, n$, be $n$ solutions to the vector differential equation

$$\dot{x} = A(t)x(t).$$

Then for the $n \times n$ matrix of solutions

$$X(t) = [x_1(t), x_2(t), \ldots, x_n(t)],$$

we have

$$\det X(t) = C \exp \left( \int \text{tr}A(t) \, dt \right), \quad \text{tr}A(t) = a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t),$$

with some constant $C$. Therefore, the Wronskian $W(t) = \det X(t)$ is either never zero, if $C \neq 0$, or else identically zero, if $C = 0$.

**Corollary:** Let $x_1(t), x_2(t), \ldots, x_n(t)$ be solutions of the homogeneous equation $\dot{x} = A(t)x$ on some interval $[a, b]$. Then the corresponding fundamental matrix $X(t) = [x_1(t), x_2(t), \ldots, x_n(t)]$ is either a singular matrix for all $t \in [a, b]$ or else non-singular. In other words, $\det X(t)$ is either identically zero or never vanishes on the interval $[a, b]$.

**Corollary:** Let $A(t)$ be an $n \times n$ matrix function that is continuous on an interval $(a, b)$. If \{ $x_1(t), x_2(t), \ldots, x_n(t)$ \} is a linearly independent set of solutions to the homogeneous differential equation $x' = Ax$ on $[a, b]$, then the Wronskian

$$W(t) = \det [x_1(t), x_2(t), \ldots, x_n(t)]$$

is not zero at every point $t$ in $[a, b]$.

**Theorem:** Let $X(t)$ be a fundamental matrix for the homogeneous linear system $\dot{x} = A(t)x$. Then the unique solution of the initial value problem

$$\dot{x} = A(t)x, \quad x(t_0) = x_0,$$

is given by

$$x(t) = X(t)X^{-1}(t_0)x_0.$$

**Corollary:** The matrix

$$\Phi(t) = X(t)X^{-1}(t_0)$$

is the solution of the following matrix initial value problem:

$$\frac{d\Phi(t)}{dt} = A(t)\Phi(t), \quad \Phi(t_0) = I,$$  \hspace{1cm} (8.12)
where \( I \) is the identity matrix. Moreover, \( \Phi(t) \) is the fundamental matrix of Eq. (8.7).

**Corollary:** Let \( x_k(t), k = 1, 2, \ldots, n, \) be solutions to the initial value problems

\[
\frac{dx_k}{dt} = A(t)x_k(t), \quad x_k(t_0) = e_k, \quad k = 1, 2, \ldots, n,
\]

where

\[
e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \ldots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.
\]

Then \( x_k(t), k = 1, 2, \ldots, n, \) form a fundamental set of solutions of the system \( x' = A(t)x \). Moreover, the matrix formed from these solutions satisfies the IVP (8.12).

**Example 8.1** The vector functions

\[
x_1 = \begin{bmatrix} t^2 \\ t \end{bmatrix}, \quad x_2 = \begin{bmatrix} t^3 \\ 0 \end{bmatrix}
\]

are two linearly independent solutions to the following homogeneous vector differential equation

\[
x(t) = A(t)x, \quad A = \frac{1}{t} \begin{bmatrix} 3 & -t \\ 0 & 1 \end{bmatrix}.
\]

The corresponding fundamental matrix is

\[
X(t) = \begin{bmatrix} t^2 & t^3 \\ t & 0 \end{bmatrix}.
\]

The trace of the matrix \( A \) is \( \text{tr} A = 4/t \), so according to the Abel theorem, the wronskian is

\[
det X = -t^4 = C e^{4 \ln t}, \quad \text{where} \quad C = -1.
\]

Now we consider the case when the vector differential equation (8.7) has constant coefficient matrix, namely,

\[
\frac{dx}{dt} = Ax(t), \quad t \in [a, b], \quad (8.13)
\]

where \( A \) is a constant square matrix.

**Theorem:** Let \( A \) be an \( n \)-by-\( n \) matrix with constant entries and let \( x_k(t) \) be the \( k \)th column of the exponential matrix \( e^{At} \). Then the vector functions \( x_1(t), x_2(t), \ldots, x_n(t) \) are linearly independent for every \( t \).

**Theorem:** Let \( X(t) = [x_1(t), x_2(t), \ldots, x_n(t)] \) be a fundamental solution set for the vector differential equation (8.13) with constant square matrix \( A \). Then

\[
e^{A(t-t_0)} = X(t)X^{-1}(t_0).
\]
Theorem: For any constant \( n \times n \) matrix \( A \), the vector-column function
\[
x(t) = e^{At} c = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n
\]
is the general solution of the linear vector differential equation \( \dot{x} = Ax(t) \). Here \( x_k(t) \) is \( k \)th column of the exponential matrix \( \Phi(t) = e^{At} \) and \( c = \langle c_1, c_2, \ldots, c_n \rangle^T \) is a constant column vector. Moreover, the vector column
\[
x(t) = e^{At} x_0
\]
is the unique solution of the initial value problem:
\[
\frac{dx}{dt} = Ax(t), \quad x(t_0) = x_0.
\]

Example 8.2 Consider the vector differential equation
\[
\dot{x} = Ax(t), \quad \text{where } A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]
The fundamental matrix for this vector equation is
\[
\Phi(t) = e^{At} = e^{2t} \begin{bmatrix} 1 + t & -t \\ t & 1 - t \end{bmatrix}.
\]
Obviously \( \Phi(0) = I \), the identity matrix, and its inverse is \( \Phi^{-1} = e^{-2t} \begin{bmatrix} 1 - t & t \\ -t & 1 + t \end{bmatrix} \). The solution to the initial value problem
\[
\dot{x} = Ax(t), \quad x(1) = (3, 2)^T,
\]
is
\[
x(t) = e^{A(t-1)} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = e^{2(t-2)} \begin{bmatrix} t & 1 - t \\ t - 1 & 2 - t \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = e^{2t-2} \begin{bmatrix} t + 2 \\ t + 1 \end{bmatrix}
\]
because
\[
\dot{x}(t) = e^{2t-2} \begin{bmatrix} 2t + 5 \\ 2t + 3 \end{bmatrix} \quad \text{and} \quad Ax = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} e^{2t-2} \begin{bmatrix} t + 2 \\ t + 1 \end{bmatrix} = e^{2t-2} \begin{bmatrix} 2t + 5 \\ 2t + 3 \end{bmatrix}.
\]