15.1 Fourier Series

In this section, we present a remarkable result that gives the mathematical explanation for how cellular phones work: Fourier series. It was discovered in the beginning of nineteenth century by Joseph Fourier (1768 – 1830) that an “arbitrary function” \( f(x) \) defined on the interval of length \( 2L \) can be represented by a convergent trigonometric series:

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right). \tag{15.1.1}
\]

Eq. (15.1.1) means that at every point \( x \) in the interval \([-L, L]\), the function \( f(x) \) is the limit of the partial sums:

\[
f(x) = \lim_{N \to \infty} S_N(x),
\]

where

\[
S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right). \tag{15.1.2}
\]

Note that for some points, the limit (15.1.2) may not exist, or when it exists, it may not equal to the value of the function \( f(x) \) at that point. Necessary and sufficient conditions for a function to be represented by a Fourier series are still waiting to be discovered. One sufficient condition (not the most general one) is presented in the following statement.

**Theorem 15.1.** Suppose that a periodic function \( f \) with a period \( 2L \) is a piecewise continuous function as is its first derivative \( f' \). Then the function \( f \) is represented by a convergent series on the interval \([-L, L]\), the Fourier series (15.1.1), whose coefficients are given by

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \quad n = 0, 1, 2, \ldots; \tag{15.1.3}
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, \ldots. \tag{15.1.4}
\]

The Fourier series converges to \( f(x) \) at all points where \( f \) is continuous and to

\[
\frac{1}{2} [f(x + 0) + f(x - 0)] = \lim_{\epsilon \to 0} \frac{1}{2} [f(x + \epsilon) + f(x - \epsilon)]
\]

at all points where \( f \) is discontinuous.
For uniform convergence of the Fourier series, we need to impose an additional condition on the function.

**Theorem 15.2.** Let \( f \) be a continuous function on \((-\infty, \infty)\) and periodic with a period of \(2L\). If \( f' \) is a piecewise continuous function on \([-L, L]\), then the Fourier series (15.1.1) converges uniformly to \( f \) on \([-L, L]\) and hence on any interval. That is, for every \( \varepsilon > 0 \), there exists an integer \( N_0 = N_0(\varepsilon) \) that depends on \( \varepsilon \) such that the partial sums

\[
|S_N(x)| = \left|\frac{a_0}{2} + \sum_{n=1}^{N} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right| < \varepsilon,
\]

for all \( N \geq N_0 \), and all \( x \in (-\infty, \infty) \).

The series (15.1.1) thereby defines a periodic function, but it may not be differentiable, or even continuous. Recall that a function on an interval \([-L, L]\) is piecewise continuous if the interval can be partitioned by a finite number of subintervals so that on each of it \( f \) is continuous, and approaches finite limits at the end points. So \( \tan x \) is not a piecewise continuous function because it has infinite jumps at points \( k\pi + \pi/2, \ k = 0, \pm 1, \pm 2, \ldots \).

The American mathematician Josiah Willard Gibbs (1839 – 1903) observed in 1898 that near points of discontinuity of \( f(x) \), the \( N \)th partial sums \( S_N(x) \) of the Fourier series for \( f \) may overshoot/undershoot by approximately 9% of the jump, regardless of the number of terms, \( N \). This is the **Gibbs phenomenon**, which was first noticed and analyzed by an English mathematician Henry Wilbraham (1825 – 1883) in 1848.

### 15.2 Even and Odd Functions

Recall that \( f \) is called an **even function** if its domain contains the point \(-x\) whenever it contains the point \( x \), and if

\[
f(-x) = f(x)
\]

for each \( x \) in the domain of \( f \). Similarly, \( f \) is said to be an **odd function** if its domain contains the point \(-x\) whenever it contains the point \( x \), and if

\[
f(-x) = -f(x)
\]

for each \( x \) in the domain of \( f \).

Any function that is a linear combination of monomials \( x^p \) with even (odd) powers \( p \) is an even (odd) function. Since Taylor’s series for cosine function \( \cos x = \sum_{k \geq 0} (-1)^k \frac{x^{2k}}{(2k)!} \) contains only even powers, it is an even function. Similarly, the sine function \( \sin x = \sum_{k \geq 0} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \) is an example of an odd function. A sum or difference of two or more even functions is an even function; for instance, \( x^2 + 1 - \cos x \) is an even function. On the other hand, a product of two even or odd functions is an even function, while a product of an even and an odd function is an odd function. For instance, \( \sin^2 x \) and \( \cos^2 x \) are even functions, while \( \sin x \cos x \) is an odd function.
When an odd function is represented by a Fourier series, its expansion will be a sine Fourier series, so all coefficients of cosine terms are zeroes: \( a_0 = 0 \) and \( a_n = 0 \) for all \( n \); hence,

\[
f(x) = \sum_{n \geq 1} b_k \sin \frac{k\pi x}{L},
\]

where

\[
b_k = \frac{2}{L} \int_0^L f(x) \sin \frac{k\pi x}{L} \, dx = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{k\pi x}{L} \, dx, \quad k = 1, 2, \ldots.
\]

We refer to this series (15.1.1) as a **Fourier sine series**. It can be considered as a series for the function \( f(x) \) with a domain of the interval \([0, L]\), and extended in odd manner to the interval \([-L, 0]\) (that is, \( f(-x) = -f(x) \)). Similarly, if a function \( g(x) \) is an even function on an interval \([-L, L]\), then its Fourier series contains only cosine functions. Therefore such series is called **Fourier cosine series** (all coefficients \( b_n \) in Eq. (15.1.1) are zeroes):

\[
g(x) = \frac{a_0}{2} + \sum_{n \geq 1} a_n \sin \frac{n\pi x}{L},
\]

where

\[
a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx, \quad n = 0, 1, 2, \ldots.
\]

The same function can be extended in either odd way or even way on the interval \([-L, 0]\); so the same function may have different Fourier series representation.

**Example 15.1.** Consider the function \( f(x) = x^2 \) on the interval \([0, 2]\). First, we extend it in an even way (see Fig. 1(a)), which leads to the Fourier cosine series:

\[
x^2 = \frac{a_0}{2} + \sum_{n \geq 1} a_n \cos \frac{n\pi x}{2},
\]

where

\[
a_0 = \int_0^2 x^2 \, dx = \frac{8}{3}, \quad a_n = \int_0^2 x^2 \cos \frac{n\pi x}{2} \, dx = \frac{16}{n^2\pi^2} \cos(n\pi) = \frac{16}{n^2\pi^2} (-1)^n.
\]

This yields the following cosine series:

\[
x^2 = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n \geq 1} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{2}.
\]

For \( N > 0 \), its partial sum approximations are

\[
x^2 \sim C_N(x) = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^N \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{2}.
\]
Now we extend the function $x^2$ into negative semi-axis in odd way (see Fig. 1(b)), which leads to the sine Fourier series:

$$x^2 = \sum_{n \geq 1} b_n \sin \frac{n \pi x}{2},$$

where

$$b_n = \int_0^2 x^2 \sin \frac{n \pi x}{2} \, dx = \frac{16}{n^3 \pi^3} \left( (-1)^n - 1 \right) - \frac{8}{n \pi} (-1)^n.$$ 

Therefore,

$$x^2 = -\frac{8}{\pi} \sum_{n \geq 1} \frac{(-1)^n}{n} \sin \frac{n \pi x}{2} - \frac{32}{\pi^3} \sum_{k=0}^{(N-1)/2} \frac{1}{(2k+1)^3} \sin \frac{(2k+1)\pi x}{2},$$

and its $N$th partial sums become (after using only odd indices: $n = 2k + 1$ in the latter sum)

$$x^2 \sim S_N(x) = -\frac{8}{\pi} \sum_{n=1}^{N} \frac{(-1)^n}{n} \sin \frac{n \pi x}{2} - \frac{32}{\pi^3} \sum_{k=0}^{(N-1)/2} \frac{1}{(2k+1)^3} \sin \frac{(2k+1)\pi x}{2}.$$ 

For the periodic extension (see Fig 1(c)) with half period $L = 1$, we have the general Fourier series:

$$x^2 = \frac{A_0}{2} + \sum_{n \geq 1} \left[ A_n \cos(n \pi x) + B_n \sin(n \pi x) \right],$$
where

\[ A_0 = \int_{-1}^{0} (x + 2)^2 \, dx + \int_{0}^{1} x^2 \, dx = \int_{0}^{2} x^2 \, dx = \frac{8}{3}, \]

\[ A_n = \int_{-1}^{0} (x + 2)^2 \cos(n\pi x) \, dx + \int_{0}^{1} x^2 \cos(n\pi x) \, dx = \frac{4}{n^2\pi^2}, \]

\[ B_n = \int_{-1}^{0} (x + 2)^2 \sin(n\pi x) \, dx + \int_{0}^{1} x^2 \sin(n\pi x) \, dx = -\frac{4}{n\pi}. \]

This leads to

\[ x^2 \sim \frac{4}{3} + \frac{4}{\pi^2} \sum_{n=1}^{N} \frac{1}{n^2} \cos(n\pi x) - \frac{4}{\pi} \sum_{n=1}^{N} \frac{1}{n} \sin(n\pi x). \]

To estimate the quality of such approximations, let us calculate the partial sums for different values of \( N \) at \( x = 2 \):

<table>
<thead>
<tr>
<th>( N )</th>
<th>sin-series</th>
<th>cos-series</th>
<th>Fourier series</th>
<th>Exact value</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0</td>
<td>3.84572</td>
<td>1.96143</td>
<td>4</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
<td>3.92094</td>
<td>1.98023</td>
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</tr>
<tr>
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<td>0</td>
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<td>1.99597</td>
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<td>1.99997</td>
<td>4</td>
</tr>
</tbody>
</table>

At the point \( x = 1 \) (where the given function \( x^2 \) is continuous), we have

<table>
<thead>
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<th>sin-series</th>
<th>cos-series</th>
<th>Fourier series</th>
<th>Exact value</th>
</tr>
</thead>
<tbody>
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<td>1.00183</td>
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<tr>
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<td>1.00048</td>
<td>1</td>
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<td>1.00008</td>
<td>1.00002</td>
<td>1</td>
</tr>
</tbody>
</table>

The Fourier series partial sums (except cosine series because even extension of \( x^2 \) is a continuous function) demonstrate the Gibbs phenomenon near the points of discontinuity \( x = 0 \) and \( x = 2 \). For instance, at \( x = 1.9 \), we have

<table>
<thead>
<tr>
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<th>cos-series</th>
<th>Fourier series</th>
<th>Exact value</th>
</tr>
</thead>
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