Let us consider a von Bertalanffy (1901 – 1972) growth model that is used since 1932 to predict evolution of the cell and the organism as a whole. From the physical point of view, the characteristic state of the living organism is that of an open system. A system is closed if no material enters or leaves it; it is said to be open if there is import and export and, therefore, change of the components. Living systems are open systems, maintaining themselves in exchange of materials with environment, and in continuous building up and breaking down of their components. However, cells and organisms cannot be in true equilibrium, but in a steady state.

Let us start with a simple model of one cell that is assumed to be in the shape of a sphere of radius \( y \), which is a function of time \( t \). Its rate of change is influenced by the amount of nutrients, which occurs uniformly over the surface area of the cell, and the respiration of waste generated, which is proportional to the volume of the cell. Since the face area of the ball is \( 4\pi y^2 \) and its volume is \( 4\pi y^3/3 \), we model the spherical cell with the differential equation:

\[
\dot{y} = ay^2 - by^3,
\]

where \( a \) and \( b \) are positive real numbers. Taking for concreteness \( a = 2 \) and \( b = 1 \), we see that its slope function \( f(y) = y^2(2 - y) \) has two equilibrium points: \( y = 0 \) and \( y = 2 \). Separation of variables yields the general solution:

\[
\int \frac{dy}{y^2(2 - y)} = \frac{1}{4} \ln \left| \frac{y}{y - 2} \right| - \frac{1}{2y} = t + \ln C \quad \Rightarrow \quad \frac{y}{y - 2} e^{-2/y} = C e^{4t}.
\]

Using Maple’s command

```
DEplot(diff(y(t),t)= 2*y(t)*y(t)-(y(t))^3 , y(t),t=0..5,
[y(1)=1,y(-1)=-1,y(0)=0,y(1)=3],color=black,arrows=medium,linecolor=blue)
```

we plot the direction field and some solutions. So \( y = 0 \) is a semi-stable critical point, and \( y = 2 \) is a stable equilibrium solution. To analyze the behavior of solutions near critical point \( y = 2 \), we linearize the slope function in its vicinity. Since the derivative of \( f'(y) = 2y^2 - y^3 \) is \( f'(y) = 4y - 3y^2 \), we have \( f'(2) = -4 \). Therefore, we expect that the solutions

\[
y(t) = 2 + Ce^{-4t}.
\]

to the linearized equation

\[
y' = -4(y - 2) = 8 - 4y
\]

will be closed to the original one: \( \frac{y - 2}{y} e^{2/y} = Ce^{-4t} \). The exponential term \( Ce^{-4t} \) indicates that \( y = 2 \) is asymptotically stable as \( t \to +\infty \). On the other hand, it is impossible to linearize the von
Bertalanffy equation in a neighborhood of $y = 0$ because it is essentially non-linear. The most we can achieve is to disregard the cubic term and consider the equation

$$y' = 2y^2.$$  

However, as Fig. 2 shows, its solution is close to the solution of the original equation only in a vicinity of the point $t = 0$ (since both solutions share the same slope at $t = 0$); in other points this approximation appears no resemblance to the true solution of the von Bertalanffy model. Here is the Mathematica code:

```mathematica
sol10 = DSolve[{y'[t] == 2*(y[t])^2, y[0] == 1}, y[t], t]
pt1 = Plot[{Evaluate[y[t] /. sol1], y[t] /. sol10}, {t, 0, 3}, Exclusions -> {1 - 2*t == 0}, ExclusionsStyle -> Dashed]
```

Now we turn our attention to the general case and consider an autonomous system of differential equation, which we write in compact vector form:

$$\mathbf{u} \overset{\text{def}}{=} \frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}), \quad (12.1)$$

where $\mathbf{u} = \mathbf{u}(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$ is $n$-vector column of unknown functions, and $\mathbf{F} = (F_1(\mathbf{u}), F_2(\mathbf{u}), \ldots, F_n(\mathbf{u}))^T$ is $n$-vector column of given functions that do not depend explicitly on $t$ (which we associate with time), but on $\mathbf{u}$. In a two-dimensional case, we have instead of Eq. (12.1) the following autonomous system:

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y), \quad (12.2)$$

where $f(x, y)$ and $g(x, y)$ are given smooth (at least, having continuous derivatives) functions of two variables. Of course, the autonomous system (12.2) can be rewritten as a single equation:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{g(x, y)}{f(x, y)}$$
that has independent of time direction field. A stationary point (or an equilibrium solution) of the system (12.2) is determined by solving the following system of simultaneous equations:

\[ f(x, y) = 0, \quad g(x, y) = 0, \]  

or in general, \( F(u) = 0 \). At that point all derivatives are 0, which means that the physical system does not move. In everyday situations, we observe only stable stationary points because an unstable equilibrium does not appear in practice as a slight perturbation in the system immediately dislodge the system far away from it.

Now we address the question: “What happens to solutions of an autonomous system of differential equations over the long term?” To answer this question, we use a linearization approach and approximate solutions near every critical point by solutions to the approximate linear problem.

Consider a two-dimensional case, and assume that \( x = a \) and \( y = b \) is an equilibrium solution of the system (12.2), that is, \( f(a, b) = g(a, b) = 0 \). Since we concentrate on what happens in a vicinity of that point, we expand both slope functions into Taylor’s series:

\[
\begin{align*}
&f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \text{higher-order terms} \\
g(x, y) = g(a, b) + g_x(a, b)(x - a) + g_y(a, b)(y - b) + \text{higher-order terms},
\end{align*}
\]

because \( f(a, b) = g(a, b) = 0 \). If we set \( X = x - a \) and \( Y = y - b \), we rewrite the system (12.4) in more compact form:

\[
\begin{align*}
\frac{dX}{dt} &= f_x(a, b)(x - a) + f_y(a, b)(y - b), \\
\frac{dY}{dt} &= g_x(a, b)(x - a) + g_y(a, b)(y - b)
\end{align*}
\]

The constant matrix of this system is called the Jacobian matrix, which we denote by \( J \). Then the above system of differential equations becomes

\[
\begin{bmatrix}
\frac{dX}{dt} \\
\frac{dY}{dt}
\end{bmatrix} = J \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}, \quad J = \begin{bmatrix} f_x(a, b) & f_y(a, b) \\ g_x(a, b) & g_y(a, b) \end{bmatrix}
\]

The system (12.5) is called the linearization of Eq. (12.2) at the equilibrium point \( (a, b) \). Since the latter is linear constant coefficient vector equation, its solutions can be found explicitly. However, solutions to linearized system advise only on qualitative behavior of solutions to the given autonomous system, and cannot be used for quantitative approximations outside a small vicinity of the critical point.

If the Jacobian matrix is not singular, then the constant coefficient system of equations (12.5) has the only one critical point—the origin. We will assume that this point is isolated, which means that
Figure 3: Abbreviations: N is node; IN is improper node; PN stands for proper node; SP is saddle point; SpP stands for spiral point; C is center.

there is no other stationary points in a small neighborhood of it. The non-linear system of equations (12.2) can be written as

\[ \dot{u} = Ju + h(u), \quad u = \langle X, Y \rangle^T, \]  

(12.6)

where \( h(u) \) can be considered as a perturbation. For the non-linear system (12.6) to be close to the linear one (12.5), the vector function \( h(u) \) must be small in the vicinity of \( u = 0 \):

\[
\|h(u)\|/\|u\| \rightarrow 0 \quad \text{as } u \rightarrow 0,
\]

(12.7)

In this case, we call the autonomous system (12.2) a **locally linear system** in the vicinity of the stationary point \( u = 0 \). Since critical points of a linear system of differential equations were classified previously based on the eigenvalues, we can extend their behavior on locally linear systems. When this is done, we summarize all cases in the following table.

### Example 12.1.

Consider the equation of damped pendulum:

\[
mL^2 \frac{d^2 \theta}{dt^2} + cL \frac{d\theta}{dt} + mgL \sin \theta = 0,
\]

where \( m \) is mass attached to one end of a rigid rod of length \( L \), a damping force is assumed to be proportional to the velocity, \( c \frac{d\theta}{dt} \), \( g \) is the acceleration due to gravity, and \( \theta \) is the angle of the rod from downward vertical position. Introducing new parameters \( \gamma = c/(mL) \) and \( \omega^2 = g/L \), we rewrite the pendulum equation as the system of first order differential equations:

\[
\begin{align*}
\dot{x} & = y, \\
\dot{y} & = -\omega^2 \sin x - \gamma y,
\end{align*}
\]

where \( x = \theta \) is the angle and \( y = \dot{\theta} \) is the velocity. To find the critical points, we equate the slope functions to zero:

\[
y = 0, \quad -\omega^2 \sin x - \gamma y = 0.
\]
This yields two kinds of stationary points: $(2\pi k, 0)$ and $(\pi + 2\pi k, 0)$, where $k$ is an integer, that correspond to two physical positions of the bob: downward and upward vertical positions. Taking for concreteness $\omega = 9$ and $\gamma = 1/5$, we linearize the system near downward vertical position $((2\pi k, 0))$:

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \text{where} \quad S = \begin{bmatrix} 0 & 1 \\ -9 \cos x & \frac{1}{5} \end{bmatrix}_{x=2\pi k} = \begin{bmatrix} 0 & 1 \\ -9 - \frac{1}{5} \end{bmatrix}.$$

Since the matrix $S$ has two complex conjugate eigenvalues with negative real part,

$$\lambda = -\frac{1}{10} \pm i \frac{\sqrt{899}}{10} \approx -0.1 \pm i 2.99833,$$

it is a stable spiral point. At the points $(\pi + 2\pi k, 0)$, the stability matrix becomes

$$J = \begin{bmatrix} 0 & 1 \\ -9 \cos x & \frac{1}{5} \end{bmatrix}_{x=2\pi k+\pi} = \begin{bmatrix} 0 & 1 \\ 9 + \frac{1}{5} \end{bmatrix},$$

which has eigenvalues with positive real part: $\lambda = \frac{1}{10} \pm i \frac{\sqrt{899}}{10} \approx 0.1 \pm i 2.99833$. Therefore these points $(\pi + 2\pi k, 0)$ are unstable spiral points.

**Example 12.2.** The Van der Pol oscillator was originally proposed in 1920 by the Dutch electrical engineer and physicist Balthasar van der Pol (1889 – 1959) while he was working at Philips Company (in the Netherlands). He found stable oscillations, which he called relaxation-oscillations and are now known as limit cycles, in electrical circuits employing vacuum tubes. When these circuits were driven near the limit cycle, they become entrained, i.e., the driving signal pulls the current along with it. The Van der Pol equation has a long history of being used in both the physical and biological sciences. The equation has also been utilised in seismology to model the two plates in a geological fault.

The van der Pol oscillator is an oscillator with nonlinear damping governed by the second-order differential equation:

$$\ddot{x} - \mu(1 - x^2) \dot{x} + x = 0,$$

where $x$ is the dynamical variable and $\mu > 0$ a parameter. When $\mu = 0$, the equation reduces to $\ddot{x} + x = 0$, and has the familiar general solution $x(t) = c_2 \cos t + c_2 \sin t$. Usually the term $-\mu(1 - x^2)$ in the equation should be regarded as friction or resistance, and this is the case when the coefficient $-\mu(1 - x^2)$ is positive. However, if the coefficient is negative, then we have the case of “negative resistance.” In the age of “vacuum tube” radios, the “tetrode vacuum tube” (cathode, grid, plate), was used for a power amplifier and was known to exhibit “negative resistance.”

The van der Pol equation can be written as a first order system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \mu(1 - x^2)y - x, \end{cases}$$

where $x$ represent current in a circuit, and $y$ is proportional to voltage. Equating the slope functions to zero,

$$y = 0, \quad \mu(1 - x^2)y - x = 0,$$
Van der Pol System with \( \mu = 1 \)

Figure 4: Solution of the van der Pol equation for \( \mu = 1 \) under the initial conditions \( x(0) = -1, \ x'(0) = 0 \), plotted with Mathematica.

we see the the van der Pol equation has only one critical point—the origin \((0, 0)\). The stability matrix \( S \) for the above system is

\[
\begin{bmatrix}
\frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\
\frac{\partial y}{\partial x} & \frac{\partial y}{\partial y}
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\
-2\mu x - 1 & \mu (1 - x^2) \end{bmatrix} \implies S = \begin{bmatrix} 0 & 1 \\
-1 & \mu \end{bmatrix}.
\]

The eigenvalues of the matrix \( S \) are

\[
\lambda = \mu \pm \sqrt{\mu^2 - 4}. 
\]

Since \( \mu > 0 \), the original is unstable stationary point. When \( \mu^2 > 4 \) it is improper node, while when \( \mu < 2 \) it is a spiral source. For \( \mu = 2 \) we have degenerate node. Note if \( \mu = 0 \), the eigenvalues are pure imaginary, consistent with this special case, which reduces to a simple harmonic oscillator. For \( \mu < 0 \), we get stable critical point.