1. Introduction. It is known from elementary physics that, in the absence of air friction, a ball thrown up from the ground into earth’s atmosphere with initial speed \( v_0 \) would attain a maximum altitude of \( \frac{v_0^2}{2g} \). In this case the return time is \( \frac{2v_0}{g} \), independent of the ball’s mass. Here \( g \) is the acceleration due to gravity. If the ball is thrown up from altitude \( y_0 \) (which we later assume to be zero), then the time \( T_0 \) spent traveling is given by

\[
\{ \text{travel time with no air resistance when thrown from height } y_0 \} = T_0 = \frac{v_0 + \sqrt{v_0^2 + 2y_0g}}{g}.
\]

The presence of air influences the ball’s motion: it experiences two forces acting on it—the force of gravity and the air resistance force. Let’s define the symbol \( T \) as follows:

\[
\{ \text{travel time with air resistance when thrown from height } y_0 \} = T
\]

Without air resistance, the object travels farther up than with air resistance. On the way down, without air resistance the object travels a larger distance, but it also gathers more speed. A natural question is, which travel time (with air resistance vs. no air resistance) is larger? Also, it is of interest to find the maximum altitude \( y_{\text{max}} \) of the ball, the time \( T_{\text{max}} \) to reach maximum altitude, and the time \( T_{\text{down}} \) to return back from \( y_{\text{max}} \). Therefore, \( T_{\text{max}} + T_{\text{down}} = T_{\text{total}} \) (the total time the ball spent in the air). The landing velocity is denoted by \( v_\ell \).

The aim of this project is to use a model of air resistance (provided here) to
(a) find \( T_{\text{total}} \) and \( T_0 \) for a ball that is thrown up from the ground \( (y_0 = 0) \) and with the given initial velocity \( v_0 \);
(b) find examples of predictions where \( T_{\text{total}} \leq T_0 \) or \( T_{\text{total}} \geq T_0 \);
(c) determine \( T_{\text{max}} \) and \( T_{\text{down}} \);
(d) discover which of the velocities \( v_0 \) or \( v_\ell \) is larger?

You will find in sections 4 through 8 a theoretical derivation of all the necessary equations. This is given for your information and benefit. Read it and try to understand it, but if you find a step or two you do not understand, you can still do this project.

Your job is to answer the questions in section 3.
Solving the equations that appear in section 3 requires the use of a computer algebra system such as Mathematica, Maple, or Maxima or a software package MATLAB. For example, in one case the equation’s unknown appears as an upper limit of an integral that cannot be calculated explicitly. While to solve such problem by hand is impossible, this is where a software package comes to the rescue.

We hope that with this project you gain an appreciation of the immense practical value of computer software packages, as well as of the powerful interplay between theoretical developments and computational approaches to problem solving.

2. A model for air resistance. Air resistance is the force that acts in the direction opposite to the motion of an object through air. Air resistance depends on the shape, material, and orientation of the object, the density of the air, and the object’s relative speed.

We would like to think that there is a nice formula for the air resistance in terms of speed and other variables. Such a formula would help in making calculations and predicting various quantities. A starting point for obtaining such a formula is our everyday experience. Based on our experience, a reasonable assumption to make is that air resistance increases in magnitude as a function of speed, and that there is no air resistance if the object has speed zero. See Figure 1.

\[ F \propto s^p, \quad \text{which may be written as} \quad F(v) = k |v|^p. \] (M)

Here \( v \) is velocity, and both \( k \) and \( p \) are positive constants. For very small objects, such as a speck of dust (about 1 micrometer or 0.001mm), \( p = 1 \) seems to give a reasonable formula for the air resistance. For larger, human scale objects moving at relatively large speed, \( p = 2 \) works better. In this project, the magnitude of the air resistance \( F \) as a function of velocity \( v \) is assumed be given by formula (M).

3. Questions. Use the equations given in section 4 to answer.

\(^1\)Our intuition based on everyday experience is limited to a small range of conditions. This may lead to erroneous assumptions.
1. A ball with mass $m = 1.0 \text{ kg}$ is thrown up from height $y_0 = 0$ with initial velocity $v_0 = 50 \text{ m/s}$. With $k = 0.01$ and $p = 2.0$, model (M) predicts $T_{\text{total}} \approx 7.212 \text{ sec}$ and $\gamma \approx 0.707$ (defined later in §4). Verify this.

2. Find $\gamma$, given that $m = 1.0$, $p = 1.5$, $k = 0.02$, $v_0 = 10.0$, and $y_0 = 0$. What does this value of $\gamma$ imply (in practical terms)?

3. $m = 1.0$, $p = 2$, $k = 0.02$, and $y_0 = 10$;
   a) Produce a plot of $v_\ell$ as a function of $v_0$, for $0 \leq v_0 \leq 10$.
   b) Produce a plot of $T$ as a function of $v_0$, for $0 \leq v_0 \leq 10$.
   c) What can be said from the graphs in parts (a) and (b)?

4. For $p = 2$, $m = 1.0$, $k = 0.02$, $v_0 = 10.0$, and $y_0 = 0$, determine $T_{\text{max}}$ and $T_{\text{down}}$.

4. Equations.

GIVEN CONSTANTS, PARAMETER VALUES, AND INITIAL VALUES.

$$g \approx 9.806 \text{ m/sec}^2$$, the acceleration due to gravity near sea level at 45 deg. latitude;

$m$ mass of the object, in kg;

$k$ drag coefficient, positive;

$p = 2$ power of the speed term in the resistance force;

$v_0$ initial velocity, positive, in m/s;

$y_0 = 0$ initial altitude, positive, in m.

QUANTITIES TO BE COMPUTED

$v_\ell$ the landing velocity with air resistance, in m/s;

$T_{\text{total}} = T_{\text{max}} + T_{\text{down}}$ the total landing time with air resistance, in seconds;

$T_0$ the landing time without air resistance, in seconds;

$\gamma$ the ratio $T_{\text{total}} / T_0$;

$T_{\text{max}}$, $T_{\text{down}}$ the time to climb to the maximum altitude $y_{\text{max}}$ and time to return.

EQUATIONS NEEDED

A) To calculate $v_\ell$ use the following equation:

$$\int_0^{v_\ell} \frac{v \, dv}{-g + \frac{k}{m} |v|^p} = -y_0 + \int_{v_0}^{0} \frac{v \, dv}{g + \frac{k}{m} |v|^p}$$ (1)

B) To calculate $T$, make sure you know $v_\ell$ first. Then use the equation
\[ T = -\int_{v_0}^{v_t} \frac{dv}{g + \frac{k}{m} |v|^p} + \int_{0}^{v_t} \frac{dv}{-g + \frac{k}{m} |v|^p} \]  

(2)

**Derivation of equations when air resistance is present**

5. Derivation of a differential equation.

The air resistance force depends on the velocity \((v)\) of the object at time \(t\), so let us denote this force with the symbol \(F(v)\). Note that the air resistance, force \(F(v)\), always acts in the direction opposite to the motion. Therefore, \(F(v)\) acts in the down (negative) direction when the ball is moving up, and it acts in the up (positive) direction when the ball is moving down. If we measure the displacement \(y = y(t)\) vertically upwards from the ground, then \(v = \frac{dy}{dt} = \dot{y}\) is the velocity of the object. Newton’s law of motion for the ball on the way up gives the differential equation

\[ m \dot{v} = -mg - F(v), \quad \text{or} \quad \frac{dv}{dt} = -g - \frac{1}{m} F(v), \]  

(3)

and on the way down,

\[ m \dot{v} = -mg + F(v), \quad \text{or} \quad \frac{dv}{dt} = -g + \frac{1}{m} F(v). \]  

(4)

Since we assume \(F(v) = k |v|^p\), the equation of motion on way up becomes

\[ m \dot{v} = -mg - k |v|^p \quad \text{or} \quad \frac{dv}{dt} = -g - \frac{k}{m} |v|^p, \]  

(5)

and on the way down,

\[ m \dot{v} = -mg + k |v|^p \quad \text{or} \quad \frac{dv}{dt} = -g + \frac{k}{m} |v|^p. \]  

(6)

6. The travel time \(T\) in terms of the landing velocity. To find an equation for \(v_t\), the landing velocity, we rewrite Eq. (5) as \(dt = -dv/(g + F(v)/m)\) and integrate both sides from \(t = 0\) and \(v = v_0\) to \(t = T_{\text{max}}\) and \(v = 0\). Here \(T_{\text{max}}\) is the time to reach the maximum altitude \(y_{\text{max}}\), which is also the time to have velocity \(v = 0\). We obtain,

\[ \int_{0}^{T_{\text{max}}} dt = -\int_{v_0}^{0} \frac{dv}{g + \frac{k}{m} |v|^p}, \quad \text{that is,} \quad T_{\text{max}} = -\int_{v_0}^{0} \frac{dv}{g + \frac{k}{m} |v|^p} \]  

(7)

A similar formula is valid for time \(T\). From Eq. (6), we get

\[ \int_{T_{\text{max}}}^{T} dt = \int_{0}^{v_t} \frac{dv}{-g + \frac{k}{m} |v|^p} \quad \Rightarrow \quad T - T_{\text{max}} = \int_{0}^{v_t} \frac{dv}{-g + \frac{k}{m} |v|^p} \]  

(8)
Equating the time $T_{\text{max}}$ in equations (7) and (8), we have

$$T = -\int_{v_0}^{0} \frac{d\nu}{g + k m |\nu|^p} + \int_{0}^{v_{\ell}} \frac{d\nu}{-g + k m |\nu|^p}$$  \hspace{1cm} (9)$$

Equation (9) gives $T$ as a function of $v_{\ell}$. The next step is to find an equation for $v_{\ell}$.

7. An equation for the landing velocity. To find an equation for $v_{\ell}$, we rewrite Eq. (5) as $v \, dt = -v \, dv / (g + F(v) / m)$ and integrate both sides from $t = 0$ and $v = v_0$ to $t = T_{\text{max}}$ and $v = 0$. Here $T_{\text{max}}$ is the time to reach the maximum altitude $y_{\text{max}}$, which is also the time to have velocity $v = 0$. Using the fact that the integral of the velocity is the displacement, we obtain,

$$\int_{0}^{T_{\text{max}}} v \, dt = -\int_{v_0}^{0} \frac{v \, dv}{g + k m |v|^p}, \quad \text{that is,} \quad y_{\text{max}} - y_0 = -\int_{v_0}^{0} \frac{v \, dv}{g + k m |v|^p}$$  \hspace{1cm} (10)$$

A similar formula is valid for the distance traveled down. From Eq. (6) we get

$$\int_{T_{\text{max}}}^{T} v \, dt = \int_{v_0}^{v_{\ell}} \frac{v \, dv}{-g + k m |v|^p} \implies -y_{\text{max}} = \int_{v_0}^{v_{\ell}} \frac{v \, dv}{-g + k m |v|^p}$$  \hspace{1cm} (11)$$

Equating the maximum distance $y_{\text{max}}$ traveled in both ways (equations (10) and (11)), we get an equation involving the landing velocity $v_{\ell}$:

$$\int_{0}^{v_{\ell}} \frac{v \, dv}{-g + k m |v|^p} = -y_0 + \int_{v_0}^{0} \frac{v \, dv}{g + k m |v|^p}$$  \hspace{1cm} (12)$$

Equation (12) is an equation where the unknown is $v_{\ell}$, which does not appear explicitly solved for.