NAME: 

The project requires the use of your lovely software (either commercial as Matlab, Maple, or Mathematica, or free software—Maxima or Octave). Hand in a hard copy of your project along with codes and printed pictures or submit your codes in plain text to <Vladimir_Dobrushkin@brown.edu>. Be sure that your name is printed, no hand writing.

The objective of this project is to learn three special functions that are very important in applications. Consider three homogeneous differential equations subject to appropriate initial conditions:

**Bessel**: 
\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - 1) y = 0, \quad y(0) = 0, \quad y'(0) = \frac{1}{2}, \quad 0 < x < \infty; \]  

**Legendre**: 
\[ \left(1 - x^2\right) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 30 y = 0, \quad y(0) = 0, \quad y'(0) = \frac{15}{8}, \quad -1 < x < 1; \]  

**Chebyshev**: 
\[ \left(1 - x^2\right) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 35 y = 0, \quad y(0) = 0, \quad y'(0) = 6, \quad -1 < x < 1. \]  

Assume that every solution of these equations can be represented as a convergent power series:

\[ y(x) = \sum_{n \geq 0} c_n x^{2n+1}. \]  

Upon substituting the series (4) into each differential equation and equating coefficients of like powers, you should obtain a relation between their coefficients, \(c_n\), called the recurrence or difference equation. Write explicitly this recurrence for each differential equation (1) – (3).

The solution of the initial value problem (1) is called the **Bessel function of the first kind** and denoted by \(J_1(x)\). This function is represented as infinite power series (4). Your task is to determine explicitly only three-term polynomial approximation

\[ \phi_5(x) = \sum_{n=0}^{2} c_n x^{2n+1}. \]  

Recall that every second order linear differential equation

\[ y'' + p(x) y' + q(x) y = 0 \]

on an interval \(I\) has two linearly independent solutions. If one solution \(y_1(x)\) is known, another one can be obtained from the formula

\[ y_2(x) = y_1(x) \int y_1^{-2}(x) \exp \left\{ - \int_{x_0}^{x} p(t) \, dt \right\} \, dx, \quad y_1(x) \neq 0 \text{ on } I. \]
Let $\psi_5(x)$ be the function obtained from Eq. (5) upon substituting $\phi_5(x)$ instead of $y_1$. Find a formula for $\psi_5(x)$ and plot three functions $J_1(x)$, $\phi_5(x)$, and $\psi_5(x)$ in the interval $(\epsilon, 6)$, for $\epsilon = 0.1$.

A polynomial solution of the initial value problem (2) is called the *Legendre polynomial* and denoted by $P_5(x)$. Find explicit expression of $P_5(x)$ and apply Eq. (5) to determine another linearly independent solution $Q_5(x)$ of the Legendre equation (2). Then plot these two functions, $P_5(x)$ and $Q_5(x)$ in the interval $-1 < x < 1$.

A polynomial solution of the initial value problem (3) is called the *Chebyshev polynomial of the second kind* and denoted by $U_5(x)$. Using Eq. (5), find another linearly independent solution $\psi_5(x)$ and plot it along with $U_5(x)$. 