Numerical Methods

Due April 24, 2006

In general, we are using numerical methods to approximate the solution to

\[ u_t = f(t, u(t)). \]

1. Using Euler’s method

\[ u_{j+1} = u_j + \Delta tf(u_j, t_j) \]

explain the concept of local error (LTE) and global error, as well as the term "order of the method".

2. What is a one-step method? Show that for a one-step method, global error is of the same order as local error.

3. Use

(a) Euler’s method
(b) Second order Runge-Kutta
(c) Fourth order Runge Kutta
(d) any multi-step method you choose
to approximate the solution to the differential equation

\[ u_t = \cos(\pi t) + u(t) \]

with initial condition \( u(0) = 2 \).

Solve this for time \( t \) up to 2.0.

Show a graph of the solution vs. the exact solution, and a graph of the errors for number of points \( N = 10, 20, 40, 80, 160, 320, 640 \).

Show the effect of increasing the number of points on the error (using the different norms. What can you conclude about the order of the method? Comment on the behavior you see and determine which method performed best.
4. Use
   (a) Euler’s method
   (b) Implicit Euler’s method
   (c) Second order Runge-Kutta
   (d) Fourth order Runge Kutta
   (e) any explicit multi-step method you choose
   (f) any implicit multi-step method you choose

to approximate the solution to the differential equation

\[ u_t = -100 \cdot u(t) \]

with initial condition

\[ u(0) = .02 \]

Solve this for time \( t \) up to 1.0, or as long as you still get reasonable answers.

Show a graph of the solution vs. the exact solution, and a graph of the errors for number of points \( N = 10, 20, 40, 80, 160, 320, 640 \). Show the effect of increasing the number of points on the error (using the different norms. What can you conclude about the order of the method? Comment on the behavior you see and determine which method performed best.

Explain the differences between the implicit and explicit methods.

5. Use
   (a) Euler’s method
   (b) Second order Runge-Kutta
   (c) Fourth order Runge Kutta

to approximate the solution to the differential equation

\[ u_t = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} u \]

with initial condition

\[ u(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

Solve this for time \( t \) up to 2.0.

Show a graph of the solution vs. the exact solution, and a graph of the errors for number of points \( N = 10, 20, 40, 80, 160, 320, 640 \).

Show the effect of increasing the number of points on the error (using the different norms. What can you conclude about the order of the method? Comment on the behavior you see and determine which method performed best.
For each problem, I want
1. A description of the problem and numerical method
2. One table showing the errors resulting from all the different methods.
3. One graph showing the numerical solution
4. One graph showing the errors
5. A writeup explaining your results and analysis
6. A copy of your codes in the appendix