We consider an example of a 3×3 system:

$$\underline{x}' = A\underline{x}$$

where

$$A = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 3.5 & 2.5 \\ 1. & -0.5 & 2.5 \end{pmatrix}$$

- Whichever of the three approaches we take to understanding the solution, we will need to compute eigenvalues, eigenvectors, and generalized eigenvectors. So let's get it out of the way first!

  1. Find the eigenvalue(s) by computing $det(A - rI) = 0$:

  $$det(A - rI) = \begin{pmatrix} 3-r & 0 & 1 \\ -1 & 3.5-r & 2.5 \\ 1. & -0.5 & 2.5-r \end{pmatrix} = 0$$

  So we have

  $$\begin{aligned} det(A-rI) &= (3-r)\left((3.5-r)(2.5-r)+(0.5)(2.5)\right)+(0.5-(3.5-r)) \\ &= (3-r)\left((\frac{7}{2}-r)(\frac{5}{2}-r)+\frac{5}{4}\right)+(r-3) \\ &= (3-r)\left((\frac{7}{2}-r)(\frac{5}{2}-r)+\frac{5}{4}-1\right) \\ &= (3-r)\left((\frac{7}{2}-r)(\frac{5}{2}-r)+\frac{5}{4}-1\right) \\ &= (3-r)\left(\frac{35}{4}-6r+r^2+\frac{1}{4}\right) \\ &= (3-r)\left(\frac{36}{4}-6r+r^2+\frac{1}{4}\right) \\ &= (3-r)^3 \end{aligned}$$

  So my eigenvalue (of multiplicity 3) is $r = 3$.

  2. Find the eigenvector $\underline{v}$ such that $(A - rI)\underline{v} = 0$:

  $$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0.5 & 2.5 \\ 1. & -0.5 & -0.5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

  So that $v_3 = 0$, and $v_1 - \frac{1}{2}v_2 = 0$. This gives us an eigenvector of the form

  $$\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

  3. Compute the first generalized eigenvector $\underline{w}$ such that $(A - rI)\underline{w} = \underline{v}$:

  $$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0.5 & 2.5 \\ 1. & -0.5 & -0.5 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

  This gives us $w_3 = 1$, $w_1 - 0.5w_2 - 0.5 = 0$ which is $w_1 - 0.5w_2 = 0.5$. It is easy to chose

  $$\underline{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

4. Compute the second generalized eigenvector $\underline{z}$ such that $(A - rI)\underline{z} = \underline{w}$:

$$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0.5 & 2.5 \\ 1. & -0.5 & -0.5 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

which gives $z_3 = 1$, $z_1 - 0.5z_2 - 0.5 = 1$ which gives a generalized eigenvector

$$\underline{z} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

So we have obtained an eigenvalue $r = 3$ and its eigenvector, first generalized eigenvector, and second generalized eigenvector:

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \underline{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \underline{z} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

- Now let's get the solution. As usual, there are 3 approaches:

1. **Guess:** To find the first solution $\underline{x}^{(1)}$ we guess that it has the form $\underline{x}^{(1)} = \underline{v}e^{rt}$ and then plug into the equation to get: $(\underline{x}^{(1)})' = r\underline{v}e^{rt}$ so that we need

$$r\underline{v}e^{rt} = A\underline{v}e^{rt}$$

which, since the exponential is a common term which is nonzero, requires

$$A\underline{v} = r\underline{v}.$$

It is well-known that a pair $r, \underline{v}$ which satisfy this relation is called an eigenvalue-eignevector pair, and we computed it above. Unfortunately, in this case only one eigenvalue and one eigenvector are found, so we can only get one solution this way.
To find the second solution, we guess $\underline{x}^{(2)} = \underline{v}te^{rt} + \underline{w}e^{rt}$ and when we differentiate and plug into the equation this gives us

$$\underline{v}e^{rt} + \underline{v}rte^{rt} + \underline{w}re^{rt} = A\underline{v}te^{rt} + A\underline{w}e^{rt}$$

canceling the e^{rt} term and rearranging this we get

$$\underline{v} = (A\underline{v} - r\underline{v})t + (A\underline{w} - r\underline{w})$$

since r and $\underline{v}$ are an eigen-pair, this gives us

$$(A - rI)\underline{w}) = \underline{v}$$

which we solved above.
To find the third solution, we guess[1] $\underline{x}^{(3)} = \frac{1}{2}\underline{v}t^2e^{rt} + \underline{w}te^{rt} + \underline{z}e^{rt}$. Plugging this into the equation (yuck!) we get:

$$\underline{v}te^{rt} + \frac{1}{2}\underline{v}t^2re^{rt} + r\underline{w}te^{rt} + \underline{w}e^{rt} + r\underline{z}e^{rt} = \frac{1}{2}A\underline{v}t^2e^{rt} + A\underline{w}te^{rt} + A\underline{z}e^{rt}$$

and canceling and rearranging we get:

$$((A\underline{w} - r\underline{w}) - \underline{v})\,t + \frac{1}{2}(A\underline{v} - r\underline{v})t^2 + ((A - rI)\underline{z} - \underline{w}) = 0$$

[1] obviously!

the first two terms vanish because of the definitions of $\underline{v}$ and $\underline{w}$, and we have the requirement

$$(A - rI)\underline{z} = \underline{w}.$$

We have computed this above, so now we need only piece this together:

$$\underline{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} e^{3t},$$

$$\underline{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} te^{3t} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3t},$$

and

$$\underline{x}^{(3)} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \frac{1}{2}t^2 e^{3t} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} te^{3t} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{3t}.$$

Of course, to have any kind of intuition as to why we would choose these crazy guesses, we need to consider the other 2 approaches!

2. **Compute the matrix exponential solution:**
Here we know the solution is of the form $\underline{x} = e^{At}\underline{c}$ because we define the matrix exponential

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \frac{1}{5!}A^5t^5 + ...$$

which, when differentiated satisfies the differential equation $\underline{x}' = A\underline{x}$.
The problem is, then, how to compute the matrix exponential. To do so, we take advantage of the fact that we can "Jordanize" the matrix A, such that $A = SJS^{-1}$. The matrix J will look like:

$$J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

and the matrix

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

Let's check this! I did this on Matlab:

```
>> J= [ 3 1 0; 0 3 1 ; 0 0 3]

J =

     3     1     0
     0     3     1
     0     0     3

>> S= [ 1 1 1 ; 2 1 -1 ; 0 1 1]

S =

     1     1     1
     2     1    -1
     0     1     1
```

```
>> iS=inv(S)

iS =

    1.0000         0   -1.0000
   -1.0000    0.5000    1.5000
    1.0000   -0.5000   -0.5000

>> S*J*iS

ans =

    3.0000         0    1.0000
   -1.0000    3.5000    2.5000
    1.0000   -0.5000    2.5000
```

Now we use the fact that $e^{At} = Se^{Jt}S^{-1}$ and that

$$e^{Jt} = \begin{pmatrix} e^{3t} & te^{3t} & \frac{1}{2}t^2e^{3t} \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{pmatrix}$$

or, in Matlab

```
>> syms t
>> expm(J*t)

ans =

[          exp(3*t),        t*exp(3*t), 1/2*t^2*exp(3*t)]
[                 0,          exp(3*t),       t*exp(3*t)]
[                 0,                 0,         exp(3*t)]
```

to give us the fundamental matrix of solutions

$$\Phi(t) = e^{At} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} e^{3t} & te^{3t} & \frac{1}{2}t^2e^{3t} \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0.5 & 1.5 \\ 1 & -0.5 & -0.5 \end{pmatrix}.$$

In Matlab I could get this by:

```
>> S*expm(J*t)*iS

ans =

[  exp(3*t)+1/2*t^2*exp(3*t),   -1/4*t^2*exp(3*t),     t*exp(3*t)-1/4*t^2*exp(3*t)]
[ -t*exp(3*t)+t^2*exp(3*t), 1/2*t*exp(3*t)+exp(3*t)-1/2*t^2*exp(3*t),5/2*t*exp(3*t)-1/2*t^2
[  t*exp(3*t),    -1/2*t*exp(3*t),    exp(3*t)-1/2*t*exp(3*t)]
```

Or, of course, just by:

```
>> A= S*J*iS

A =
```

```
    3.0000         0    1.0000
   -1.0000    3.5000    2.5000
    1.0000   -0.5000    2.5000

>> expm(A*t)

ans =

[ exp(3*t)+1/2*t^2*exp(3*t),     -1/4*t^2*exp(3*t),   t*exp(3*t)-1/4*t^2*exp(3*t)]
[ -t*exp(3*t)+t^2*exp(3*t),1/2*t*exp(3*t)+exp(3*t)-1/2*t^2*exp(3*t),5/2*t*exp(3*t)-1/2*t^2*
[    t*exp(3*t),    -1/2*t*exp(3*t),      exp(3*t)-1/2*t*exp(3*t)]
```

Now, recall that our solution is $\underline{x} = e^{At}\underline{c}$ so that we have $\underline{x} = Se^{Jt}S^{-1}\underline{c}$. Since $S^{-1}\underline{c}$ is just a bunch of constants, a good fundamental matrix of solutions is $\Psi(t) = Se^{Jt}$ or

```
>> S*expm(J*t)

ans =

[ exp(3*t), t*exp(3*t)+exp(3*t), 1/2*t^2*exp(3*t)+t*exp(3*t)+exp(3*t)]
[2*exp(3*t),  2*t*exp(3*t)+exp(3*t),     t^2*exp(3*t)+t*exp(3*t)-exp(3*t)]
[0,    exp(3*t),   t*exp(3*t)+exp(3*t)]
```

3. **Partially decouple the system and solve it:**
In this approach we also use the matrices found above to give us the information that $A = SJS^{-1}$. However, we do not yet require the exact form of the matrix S, just the knowledge that this matrix can be transformed using S into a Jordan block matrix. So now we use this transformation:
$$\underline{x} = S\underline{y}$$
this means that
$$\underline{x}' = S\underline{y}'$$
which (plugging back into the ODE) gives us
$$S\underline{y}' = AS\underline{y}.$$
Left-multiply both sides by S^{-1} and use the fact that $J = S^{-1}AS$ to give the ODE
$$\underline{y}' = J\underline{y}.$$
Now let's write out this ODE and solve[2]:
$$\begin{aligned} y_1' &= 3y_1 + y_2 \\ y_2' &= 3y_2 + y_3 \\ y_3' &= 3y_3 \end{aligned}$$
The last equation is easy to solve: $y_3 = k_3 e^{3t}$. Now we can plug this into the second equation, which becomes:
$$\begin{aligned} y_2' &= 3y_2 + k_3 e^{3t} \\ y_2' - 3y_2 &= k_3 e^{3t} \end{aligned}$$

[2]Note that at this point we could solve this ODE using e^{Jt} and the fact that we know to compute it, and then $\underline{x} = Se^{Jt}\underline{k}$ and we'd be done, but that would not be as much fun.

$$
\begin{aligned}
e^{-3t}y_2' - 3e^{-3t}y_2 &= k_3 \\
\left(e^{-3t}y_2\right)' &= k_3 \\
\left(e^{-3t}y_2\right) &= k_3 t + k_2 \\
y_2 &= k_3 t e^{3t} + k_2 e^{3t}.
\end{aligned}
$$

We now plug this solution into the first equation to obtain:

$$
\begin{aligned}
y_1' &= 3y_1 + k_3 t e^{3t} + k_2 e^{3t} \\
y_1' e^{-3t} - 3e^{-3t} y_1 + &= k_3 t + k_2 \\
\left(e^{-3t}y_1\right)' &= k_3 t + k_2 \\
\left(e^{-3t}y_1\right)' &= \frac{1}{2}k_3 t^2 + k_2 t + k_1 \\
y_1 &= \frac{1}{2}k_3 t^2 e^{3t} + k_2 t e^{3t} + k_1 e^{3t}.
\end{aligned}
$$

So our solution is:

$$
\underline{y} = \begin{pmatrix} \frac{1}{2}k_3 t^2 e^{3t} + k_2 t e^{3t} + k_1 e^{3t} \\ k_3 t e^{3t} + k_2 e^{3t} \\ k_3 e^{3t} \end{pmatrix} = \begin{pmatrix} e^{3t} & te^{3t} & \frac{1}{2}t^2 e^{3t} \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}
$$

which should look familiar!