7.9.1: $x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x + \begin{pmatrix} e^t \\ t \end{pmatrix}$, $|A - rI| = (2-r)(-2-r) + 3$

$= r^2 - 1 = (r+1)(r-1) = 0$

$r_1 = 1$; $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$r_2 = -1$: $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\underline{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3u_1 = u_2 \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\underline{x}_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$

$\underline{x}_p = \underline{a} t e^t + \underline{b} e^t + \underline{c} t + \underline{d}$

$\Rightarrow \underline{a} e^t + \underline{a} t e^t + \underline{b} e^t + \underline{c} = A\underline{a} t e^t + A\underline{b} e^t + A\underline{c} t + A\underline{d} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$

$\Rightarrow \underline{a} = A\underline{a} \Rightarrow 0 = (A - I)\underline{a} = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\underline{a} \Rightarrow \underline{a} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$

$\underline{a} + \underline{b} = A\underline{b} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha - 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Rightarrow b_1 - b_2 = \alpha - 1$

$b_1 - b_2 = \frac{\alpha}{3}$

$\Rightarrow \alpha - 1 = \frac{\alpha}{3} \Rightarrow \frac{2}{3}\alpha = 1 \Rightarrow \alpha = 3/2$, $\underline{b} = \begin{pmatrix} k + 1/2 \\ k \end{pmatrix}$

Choose $k = -3/4$, so that the vector will be similar to the eigenvector for e^t, or so that we match the back of the book, or match variation of parameters

$\Rightarrow \underline{c} = A\underline{d}$, $\underline{0} = A\underline{c} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2c_1 - c_2 \\ 3c_1 - 2c_2 \end{pmatrix} \Rightarrow c_1 = 1, c_2 = 2$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2d_1 - d_2 \\ 3d_1 - 2d_2 \end{pmatrix} \Rightarrow d_1 = 0, d_2 = -1$

$\underline{x}_p(t) = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^t + -\frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^t - \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

7.9.2: $x' = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} x + \begin{pmatrix} e^t \\ \sqrt{3} e^{-t} \end{pmatrix}$ $|A - rI| = (1-r)(-1-r) - 3 = r^2 - 4$

$= (r-2)(r+2) = 0$

$r_1 = 2$; $\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

$r_2 = -2$; $\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$

$\underline{x}_h = c_1 \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{-2t}$

$\underline{x}_p = \underline{a} e^t + \underline{b} e^{-t}$

$\underline{a} e^t - \underline{b} e^{-t} = A\underline{a} e^t + A\underline{b} e^{-t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} e^{-t}$

$\underline{a} = A\underline{a} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix} = (A - I)\underline{a} \Rightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$\Rightarrow a_2 = \frac{-1}{\sqrt{3}}$, $0 = \sqrt{3} a_1 + 2 \cdot \frac{1}{\sqrt{3}} \Rightarrow a_1 = -2/3$

$-\underline{b} = A\underline{b} + \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix} = (A + I)\underline{b} = \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$-\sqrt{3} = \sqrt{3} b_1 \Rightarrow b_1 = -1$, $-0 = -2 + \sqrt{3} b_2$, $b_2 = \frac{2}{\sqrt{3}}$

$x(t) = c_1 \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{-2t} - \begin{pmatrix} 2/3 \\ 1/\sqrt{3} \end{pmatrix} e^t + \begin{pmatrix} -1 \\ 2/\sqrt{3} \end{pmatrix} e^{-t}$

7.9.3: $x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$; $|A - rI| = (2-r)(-2-r)+5$

$= r^2 + 1 = 0 \Rightarrow r = \pm i$

$r_1 = i$: $\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

$\underline{X}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} (\cos t + i \sin t) = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$

$X_h = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$

or $d_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + d_2 \begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}$

$u' = \Psi^{-1} g$; $\Psi = \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{bmatrix}$, $|\Psi| = -1$

$\Psi^{-1} = \begin{bmatrix} -\sin t & \cos t + 2\sin t \\ \cos t & \sin t - 2\cos t \end{bmatrix}$; $\Psi^{-1} g = \begin{bmatrix} \cos t \sin t + \sin t \cos t + 2\sin^2 t \\ -\cos^2 t + \sin^2 t - 2\sin t \cos t \end{bmatrix}$

$u = \int_{t_0}^{t} \Psi^{-1} g \, ds = \begin{bmatrix} -\frac{1}{2}\cos 2t + \frac{2}{2} t - \frac{2}{4} \sin 2t + c_1 \\ -\frac{1}{2} \sin 2t + \frac{1}{2} \cos 2t + c_2 \end{bmatrix}$

$x = \Psi u = \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{bmatrix} \begin{bmatrix} -\frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t + t + c_1 \\ -\frac{1}{2}\sin 2t + \frac{1}{2}\cos 2t + c_2 \end{bmatrix}$

$= \begin{bmatrix} -\cos t \cos 2t - \cos t \sin 2t + 2t\cos t + 2c_1 \cos t + \frac{1}{2}\sin t \cos 2t + \frac{1}{2}\sin t \sin 2t \\ -\frac{1}{2}\cos t \cos 2t - \frac{1}{2}\cos t \sin 2t + t\cos t + c_1 \cos t \end{bmatrix}$

$+ \begin{bmatrix} -t\sin t - c_1 \sin t - \frac{1}{2}\cos t \sin 2t + \frac{1}{2}\cos t \cos 2t + c_2 \cos t - \sin t \sin 2t \\ -\frac{1}{2}\sin t \sin 2t + \frac{1}{2}\sin t \cos 2t + c_2 \sin t \end{bmatrix}$

$+ \begin{bmatrix} +\sin t \cos 2t + 2c_2 \sin t \\ \end{bmatrix}$

$= \begin{bmatrix} -\cos^3 t + \cos t \sin^2 t - 2\sin t \cos^2 t + 2t\cos t + 2c_1 \cos t + \frac{1}{2}\sin t \cos^2 t \\ -\frac{1}{2}\cos^3 t + \frac{1}{2}\cos t \sin^2 t - \cos^2 t \sin t + t\cos t + c_1 \cos t \end{bmatrix} \checkmark$

$+ \begin{bmatrix} -\frac{1}{2}\sin^3 t + \sin^2 t \cos t - t\sin t - c_1 \sin t - \sin t \cos^2 t + \frac{1}{2}\cos^3 t \\ -\sin^2 t \cos t + \frac{1}{2}\sin t \cos^2 t - \frac{1}{2}\sin^3 t + c_2 \sin t \end{bmatrix} \checkmark$

$+ \begin{bmatrix} -\frac{1}{2}\cos t \sin^2 t + c_2 \cos t - 2\sin^2 t \cos t + \sin t \cos^2 t - \sin^3 t + 2c_2 \sin t \\ \end{bmatrix}$

$= \begin{bmatrix} -\frac{1}{2}\cos^3 t - \frac{1}{2}\cos t \sin^2 t - \frac{3}{2}\sin t \cos^2 t + 2t\cos t + 2c_1 \cos t \\ -\frac{1}{2}\cos^3 t - \frac{1}{2}\cos t \sin^2 t - \frac{1}{2}\sin t \cos^2 t - \frac{1}{2}\sin^3 t \end{bmatrix}$

$+ \begin{bmatrix} -t\sin t - c_1 \sin t + c_2 \cos t + 2c_2 \sin t - \frac{3}{2}\sin^3 t \\ + t\cos t + c_1 \cos t + c_2 \sin t \end{bmatrix}$

$= \begin{bmatrix} -\frac{1}{2}\cos t - \frac{3}{2}\sin t + 2t\cos t - t\sin t + c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t) \\ -\frac{1}{2}\cos t - \frac{1}{2}\sin t + t\cos t + c_1 \cos t + c_2 \sin t \end{bmatrix}$

7.9.3 : $x(t) = c_1\begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + c_2\begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$

$+ \begin{bmatrix} 2 \\ 1 \end{bmatrix} t\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} t\sin t - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos t - \frac{1}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix}\sin t$

$= c_1\begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + c_2\begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} t\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} t\sin t$

$- \begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos t + \frac{1}{2}\begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} - \frac{1}{2}\begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$

$= (c_1 + \tfrac{1}{2})\begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + (c_2 - \tfrac{1}{2})\begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} t\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} t\sin t$

$- \begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos t$; Let $\underline{a} = \begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix}$; $\underline{b} = \begin{bmatrix} 5\sin t \\ -\cos t + 2\sin t \end{bmatrix}$

$= (c_1 + \tfrac{1}{2}) \cdot \frac{1}{5}(2\underline{a} - \underline{b}) + (c_2 - \tfrac{1}{2}) \cdot \frac{1}{5}(\underline{a} + 2\underline{b}) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} t\cos t$

$- \begin{bmatrix} 1 \\ 0 \end{bmatrix} t\sin t - \frac{1}{\cancel{}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos t$

$= \underbrace{\frac{2c_1 + c_2 + \frac{1}{2}}{5}}_{d_1}\underline{a} + \underbrace{\frac{2c_2 - c_1 - \frac{3}{2}}{5}}_{d_2}\underline{b} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} t\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} t\sin t$

$+ \begin{bmatrix} -1 \\ -1 \end{bmatrix}\cos t$

$= d_1\begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} + d_2\begin{bmatrix} 5\sin t \\ -\cos t + 2\sin t \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} t\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} t\sin t$

$- \begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos t$

7.9.4 ; $x' = \begin{pmatrix} 1 & -1 \\ 4 & -2 \end{pmatrix} x + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$ $|A - r\underline{I}| = (1-r)(-2-r) - 4$

$= r^2 + r - 6 = (r+3)(r-2) = 0 \Rightarrow r_1 = 2, r_2 = -3$

$r_1 = 2 : \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$r_2 = -3 : \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$X_g = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-3t}$

$\Psi = \begin{pmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{pmatrix}$ $|\Psi| = -5e^{-t}$ $\Psi^{-1} = \begin{pmatrix} \frac{4}{5}e^{-2t} & \frac{1}{5}e^{-2t} \\ \frac{1}{5}e^{3t} & -\frac{1}{5}e^{3t} \end{pmatrix}$

7.9.4: $u' = \Psi^{-1} g = \begin{pmatrix} \frac{4}{5}e^{-2t} & \frac{1}{5}e^{-2t} \\ \frac{1}{5}e^{3t} & -\frac{1}{5}e^{3t} \end{pmatrix} \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix} = \begin{pmatrix} \frac{4}{5}e^{-4t} - \frac{2}{5}e^{-t} \\ \frac{1}{5}e^{3t} + \frac{2}{5}e^{4t} \end{pmatrix}$

$u = \int_{t_0}^{t} \Psi^{-1} g = \begin{pmatrix} -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} + c_1 \\ \frac{1}{5}e^{t} + \frac{1}{10}e^{4t} + c_2 \end{pmatrix}$

$x = \Psi u = \begin{pmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} + c_1 \\ \frac{1}{5}e^{t} + \frac{1}{10}e^{4t} + c_2 \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{5}e^{-2t} + \frac{2}{5}e^{t} + c_1e^{2t} + \frac{1}{5}e^{-2t} + \frac{1}{10}e^{t} + c_2e^{-3t} \\ -\frac{1}{5}e^{-2t} + \frac{2}{5}e^{t} + c_1e^{2t} - \frac{4}{5}e^{-2t} - \frac{2}{5}e^{t} - 4c_2e^{-3t} \end{pmatrix}$

$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{t}$

7.9.5: $x' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} x + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, t > 0;\ |A - rI| = (4-r)(-4-r) + 16$

$= r^2 = 0$

$r_1 = 0: \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \underline{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$r_2 = 0: \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$

$x_g = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$

$\Psi = \begin{pmatrix} 1 & t \\ 2 & 2t - \frac{1}{2} \end{pmatrix}$ $|\Psi| = -\frac{1}{2}$ $\Psi^{-1} = \begin{pmatrix} -4t+1 & 2t \\ 4 & -2 \end{pmatrix}$

$u' = \Psi^{-1} g = \begin{pmatrix} -4t+1 & 2t \\ 4 & -2 \end{pmatrix} \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix} = \begin{pmatrix} -4t^{-2} + t^{-3} - 2t^{-1} \\ 4t^{-3} \quad 2t^{-2} \end{pmatrix}$

$u = \int_{t_0}^{t} \Psi^{-1} g = \begin{pmatrix} 4t^{-1} - \frac{1}{2}t^{-2} - 2\ln t + c_1 \\ -2t^{-2} - 2t^{-1} + c_2 \end{pmatrix}$

$x = \begin{pmatrix} 1 & t \\ 2 & 2t - \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4t^{-1} - \frac{1}{2}t^{-2} - 2\ln t + c_1 \\ -2t^{-2} - 2t^{-1} + c_2 \end{pmatrix}$

$= \begin{pmatrix} 4t^{-1} - \frac{1}{2}t^{-2} - 2\ln t + c_1 - 2t^{-1} - 2 + c_2t \\ 8t^{-1} - t^{-2} - 4\ln t + 2c_1 - 4t^{-1} + t^{-2} - 4 + t^{-1} + 2c_2t - \frac{1}{2}c_2 \end{pmatrix}$

$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right] - \begin{pmatrix} 2 \\ 4 \end{pmatrix} \ln t + \begin{pmatrix} 2 \\ 5 \end{pmatrix} t^{-1} + \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} t^{-2}$

$-\begin{pmatrix} 2 \\ 4 \end{pmatrix} = (c_1 - 2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \ldots = d_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \ldots$

$7.9.6: x' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} x + \begin{pmatrix} t^{-1} \\ 2t^{-1}+4 \end{pmatrix}, \quad t>0$

$|A - rI| = (-4-r)(-1-r) - 4 = r^2 + 5r = 0$

$r_1 = 0: \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$r_2 = -5: \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$x_g = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -1 \end{pmatrix}e^{-5t}$

$T = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \quad |T| = -5 \quad T^{-1} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$

$x' = TDT^{-1}x + g(t) \Rightarrow y' = Dy + T^{-1}g, \quad T^{-1}x = y$

$y_1' = \frac{1}{5}t^{-1} + \frac{4}{5}t^{-1} + \frac{8}{5} = t^{-1} + \frac{8}{5}$

$y_2' = -5y_2 + \frac{2}{5}t^{-1} - \frac{2}{5}t^{-1} - \frac{4}{5} = -5y_2 - \frac{4}{5}$

$y_1 = \ln(t) + \frac{8}{5}t + c_1$

$y_2 = -\frac{4}{25} + c_2 e^{-5t}$

$\underline{x} = Ty = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \ln(t) + \frac{8}{5}t + c_1 \\ -\frac{4}{25} + c_2 e^{-5t} \end{pmatrix}$

$= \begin{pmatrix} \ln t + \frac{8}{5}t + c_1 - \frac{8}{25} + 2c_2 e^{-5t} \\ 2\ln t + \frac{16}{5}t + 2c_1 + \frac{4}{25} - c_2 e^{-5t} \end{pmatrix}$

$= c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -1 \end{pmatrix}e^{-5t} - \frac{4}{25}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \ln t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{8}{5}t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$7.9.7: x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x + \begin{pmatrix} 2 \\ -1 \end{pmatrix}e^t; \quad |A - rI| = (1-r)(1-r) - 4$

$= r^2 - 2r - 3 = (r-3)(r+1) = 0$

$r_1 = 3: \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$r_2 = -1: \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$x_g = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}$

$T = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \quad |T| = -4 \quad T^{-1} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix}$

$x = TDT^{-1}x + g(t) \Rightarrow y' = Dy + T^{-1}g, \quad T^{-1}x = y$

$y_1' = 3y_1 + e^t + -\frac{1}{4}e^t = 3y_1 + \frac{3}{4}e^t$

$y_2' = -y_2 + e^t + \frac{1}{4}e^t = -y_2 + \frac{5}{4}e^t$

$y_1 = -\frac{3}{8}e^t + c_1 e^{3t}; \quad Ty = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -\frac{3}{8}e^t + c_1 e^{3t} \\ \frac{5}{8}e^t + c_2 e^{-t} \end{bmatrix}$

$y_2 = \frac{5}{8}e^t + c_2 e^{-t}$

$\underline{x} = Ty = \begin{pmatrix} -\frac{3}{8}e^t + c_1 e^{3t} + \frac{5}{8}e^t + c_2 e^{-t} \\ -\frac{6}{8}e^t + 2c_1 e^{3t} - \frac{10}{8}e^t - 2c_2 e^{-t} \end{pmatrix}$

$= c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t} + \begin{pmatrix} 1/4 \\ -2 \end{pmatrix}e^t$

7.9.8: $x' = \begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix}x + \begin{pmatrix}1\\ -1\end{pmatrix}e^t$, $sX(s) - x(0) = AX(s) + G(s)$.

Assume $x(0) = 0 \Rightarrow (sI - A)X(s) = G(s)$

$(sI - A) = \begin{pmatrix}s-2 & 1\\ -3 & s+2\end{pmatrix}$ $|sI - A| = (s^2 - 4 + 3) = s^2 - 1$; $G(s) = \begin{pmatrix}\frac{1}{s-1}\\ \frac{-1}{s-1}\end{pmatrix}$

$(sI - A)^{-1} = \frac{1}{s^2-1}\begin{pmatrix}s+2 & -1\\ 3 & s-2\end{pmatrix}$; $(sI - A^{-1})G = \frac{1}{s^2-1}\begin{pmatrix}s+2 & -1\\ 3 & s-2\end{pmatrix}\begin{pmatrix}\frac{1}{s-1}\\ \frac{-1}{s-1}\end{pmatrix}$

$= \frac{1}{s^2-1}\begin{pmatrix}\frac{s+3}{s-1}\\ \frac{-s+5}{s-1}\end{pmatrix} = \begin{pmatrix}\frac{2}{(s-1)^2} + \frac{1/2}{s-1} - \frac{1/2}{s+1}\\ \frac{2}{(s-1)^2} + \frac{3/2}{s-1} - \frac{3/2}{s+1}\end{pmatrix} = X(s)$

$\mathcal{L}^{-1}(X(s)) = x(t) = \begin{pmatrix}2te^t + \frac{-1}{2}e^t + \frac{1}{2}e^{-t}\\ 2te^t + \frac{-3}{2}e^t + \frac{3}{2}e^{-t}\end{pmatrix}$

$|A - rI| = (2-r)(-2-r) + 3 = r^2 - 1 = (r+1)(r-1) = 0$

$r_1 = 1: \begin{pmatrix}1 & -1\\ 3 & -3\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix}1\\ 1\end{pmatrix}$; $r_2 = -1: \begin{pmatrix}3 & -1\\ 3 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix}1\\ 3\end{pmatrix}$

$x(t) = c_1\begin{pmatrix}1\\ 1\end{pmatrix}e^t + c_2\begin{pmatrix}1\\ 3\end{pmatrix}e^{-t} + \begin{pmatrix}1\\ 0\end{pmatrix}e^t + \begin{pmatrix}2\\ 2\end{pmatrix}te^t$

7.9.9: $x' = \begin{pmatrix}-5/4 & 3/4\\ 3/4 & -5/4\end{pmatrix}x + \begin{pmatrix}2t\\ e^t\end{pmatrix}$, $sX(s) - x(0) = AX(s) + G(s)$

Assume $x(0) = 0 \Rightarrow (sI - A)X(s) = G(s)$

$(sI - A) = \begin{pmatrix}s + 5/4 & -3/4\\ -3/4 & s + 5/4\end{pmatrix}$ $|sI - A| = s^2 + \frac{5}{2}s + 1 = (s+2)(s+\frac{1}{2})$

$(sI - A)^{-1} = \frac{1}{(s+2)(s+1/2)}\begin{pmatrix}s + 5/4 & 3/4\\ 3/4 & s + 5/4\end{pmatrix}$; $G(s) = \begin{pmatrix}\frac{2}{s^2}\\ \frac{1}{s-1}\end{pmatrix}$

$(sI - A)^{-1}G(s) = \frac{1}{(s+2)(s+1/2)}\begin{pmatrix}\frac{2s + 5/2}{s^2} + \frac{3/4}{s-1}\\ \frac{3/2}{s^2} + \frac{s + 5/4}{s-1}\end{pmatrix}$

$= \begin{pmatrix}\frac{1/4}{s+2} + \frac{4}{s+1/2} + \frac{5/2}{s^2} - \frac{17/4}{s} + \frac{1/6}{s+2} - \frac{1/3}{s+1/2} + \frac{1/6}{s-1}\\ \frac{-1/4}{s+2} + \frac{4}{s+1/2} + \frac{3/2}{s^2} - \frac{15/4}{s} - \frac{1/6}{s+2} - \frac{1/3}{s+1/2} + \frac{1/2}{s-1}\end{pmatrix}$

$= \begin{pmatrix}\frac{5/12}{s+2} + \frac{11/3}{s+1/2} + \frac{5/2}{s^2} - \frac{17/4}{s} + \frac{1/6}{s-1}\\ -\frac{5/12}{s+2} + \frac{11/3}{s+1/2} + \frac{3/2}{s^2} - \frac{15/4}{s} + \frac{1/2}{s-1}\end{pmatrix}$; $\mathcal{L}^{-1}(X(s)) = x(t) =$

$= \frac{5}{12}\begin{pmatrix}1\\ -1\end{pmatrix}e^{-2t} + \frac{11}{3}\begin{pmatrix}1\\ 1\end{pmatrix}e^{-t/2} + \frac{1}{2}\begin{pmatrix}5\\ 3\end{pmatrix}t - \frac{1}{4}\begin{pmatrix}17\\ 15\end{pmatrix} + \frac{1}{6}\begin{pmatrix}1\\ 3\end{pmatrix}e^t$

$|A - rI| = (-5/4 - r)(-5/4 - r) - \frac{9}{16} = r^2 + \frac{5}{2}r + 1 = (r + \frac{1}{2})(r + 2) = 0$

$r_1 = -\frac{1}{2}: \begin{pmatrix}-3/4 & 3/4\\ 3/4 & -3/4\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix}1\\ 1\end{pmatrix}$

$r_2 = -2: \begin{pmatrix}3/4 & 3/4\\ 3/4 & 3/4\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix}1\\ -1\end{pmatrix}$

$x(t) = c_1\begin{pmatrix}1\\ 1\end{pmatrix}e^{-t/2} + c_2\begin{pmatrix}1\\ -1\end{pmatrix}e^{-2t} + \begin{pmatrix}5/2\\ 3/2\end{pmatrix}t - \begin{pmatrix}17/4\\ 15/4\end{pmatrix} + \begin{pmatrix}1/6\\ 1/2\end{pmatrix}e^t$

7.9.10: $x' = \begin{pmatrix}-3 & \sqrt{2}\\ \sqrt{2} & -2\end{pmatrix}x + \begin{pmatrix}1\\-1\end{pmatrix}e^{-t}$; $|A-rI| = (-3-r)(-2-r)-2$

$= r^2+5r+4 = (r+1)(r+4) = 0$

$r_1 = -1: \begin{pmatrix}-2 & \sqrt{2}\\ \sqrt{2} & -1\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}$

$r_2 = -4: \begin{pmatrix}1 & \sqrt{2}\\ \sqrt{2} & 2\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \underline{v} = \begin{pmatrix}\sqrt{2}\\ -1\end{pmatrix}$

$\Psi = \begin{pmatrix}e^{-t} & \sqrt{2}e^{-4t}\\ \sqrt{2}e^{-t} & -e^{-4t}\end{pmatrix}$ $|\Psi| = -3e^{-5t}$ $\Psi^{-1} = \begin{pmatrix}\frac{1}{3}e^{t} & \frac{\sqrt{2}}{3}e^{t}\\ \frac{\sqrt{2}}{3}e^{4t} & -\frac{1}{3}e^{4t}\end{pmatrix}$

$u' = \Psi^{-1}g = \begin{pmatrix}\frac{1}{3}e^{t} & \frac{\sqrt{2}}{3}e^{t}\\ \frac{\sqrt{2}}{3}e^{4t} & -\frac{1}{3}e^{4t}\end{pmatrix}\begin{pmatrix}e^{-t}\\ -e^{-t}\end{pmatrix} = \begin{pmatrix}\frac{1}{3}-\frac{\sqrt{2}}{3}\\ \frac{\sqrt{2}}{3}e^{3t}+\frac{1}{3}e^{3t}\end{pmatrix}$

$u = \int_{t_0}^{t}\Psi^{-1}g = \begin{pmatrix}\frac{1-\sqrt{2}}{3}t + c_1\\ \frac{\sqrt{2}+1}{9}e^{3t} + c_2\end{pmatrix}$

$x = \Psi u = \begin{pmatrix}e^{-t} & \sqrt{2}e^{-4t}\\ \sqrt{2}e^{-t} & -e^{-4t}\end{pmatrix}\begin{pmatrix}\frac{1-\sqrt{2}}{3}t + c_1\\ \frac{\sqrt{2}+1}{9}e^{3t} + c_2\end{pmatrix}$

$= \begin{pmatrix}\frac{1-\sqrt{2}}{3}te^{-t} + c_1e^{-t} + \frac{2+\sqrt{2}}{9}e^{-t} + c_2\sqrt{2}e^{-4t}\\ \frac{\sqrt{2}-2}{3}te^{-t} + c_1\sqrt{2}e^{-t} - \frac{\sqrt{2}+1}{9}e^{-t} - c_2e^{-4t}\end{pmatrix}$

$= c_1\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}e^{-t} + c_2\begin{pmatrix}\sqrt{2}\\ -1\end{pmatrix}e^{-4t} + \frac{1}{3}\begin{pmatrix}1-\sqrt{2}\\ \sqrt{2}-2\end{pmatrix}te^{-t} + \frac{1}{9}\begin{pmatrix}2+\sqrt{2}\\ -\sqrt{2}-1\end{pmatrix}e^{-t}$

7.9.11: $x' = \begin{pmatrix}2 & -5\\ 1 & -2\end{pmatrix}x + \begin{pmatrix}0\\ \cos t\end{pmatrix}$, $0 < t < \pi$

$|A-rI| = (2-r)(-2-r)+5 = r^2+1 = 0 \Rightarrow r = \pm i$

$r_1 = i: \begin{pmatrix}2-i & -5\\ 1 & -2-i\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix}5\\ 2-i\end{pmatrix}$

$x_1 = (\cos t + i\sin t)\begin{pmatrix}5\\ 2-i\end{pmatrix} = \begin{pmatrix}5\cos t\\ 2\cos t + \sin t\end{pmatrix} + i\begin{pmatrix}5\sin t\\ 2\sin t - \cos t\end{pmatrix}$

$x_g = c_1\begin{pmatrix}5\cos t\\ 2\cos t + \sin t\end{pmatrix} + c_2\begin{pmatrix}5\sin t\\ 2\sin t - \cos t\end{pmatrix}$

$\Psi = \begin{bmatrix}5\cos t & 5\sin t\\ 2\cos t + \sin t & 2\sin t - \cos t\end{bmatrix}$ $|\Psi| = -5$ $\Psi^{-1} = \begin{bmatrix}-\frac{2}{5}\sin t + \frac{1}{5}\cos t & \sin t\\ \frac{2}{5}\cos t + \frac{1}{5}\sin t & -\cos t\end{bmatrix}$

$u' = \Psi^{-1}g = \begin{bmatrix}-\frac{2}{5}\sin t + \frac{1}{5}\cos t & \sin t\\ \frac{2}{5}\cos t + \frac{1}{5}\sin t & -\cos t\end{bmatrix}\begin{bmatrix}0\\ \cos t\end{bmatrix} = \begin{bmatrix}\sin t\cos t\\ -\cos^2 t\end{bmatrix}$

$\underline{u} = \int_{t_0}^{t}\Psi^{-1}g\,ds = \begin{bmatrix}\frac{1}{2}\sin^2 t + c_1\\ -\frac{t}{2} - \frac{1}{4}\sin 2\theta + c_2\end{bmatrix}$

$x = \Psi u = \begin{pmatrix}5\cos t & 5\sin t\\ 2\cos t + \sin t & 2\sin t - \cos t\end{pmatrix}\begin{bmatrix}\frac{1}{2}\sin^2 t + c_1\\ -\frac{t}{2} - \frac{1}{4}\sin 2t + c_2\end{bmatrix}$

$= \begin{bmatrix}\frac{5}{2}\cos t\sin^2 t + 5c_1\cos t - \frac{5}{2}t\sin t - \frac{5}{4}\sin t\sin 2t + 5c_2\sin t\\ \cos t\sin^2 t + \frac{1}{2}\sin^3 t + 2c_1\cos t + c_1\sin t - t\sin t + \frac{1}{2}t\cos t - \frac{1}{2}\sin t\sin 2t\end{bmatrix}$

7.9.11: $+\begin{bmatrix} 0 \\ \frac{1}{4}\cos t\sin 2t + 2c_2\sin t - c_2\cos t \end{bmatrix}$

$\frac{5}{4}\sin t\sin 2t = \frac{5}{2}\sin^2 t\cos t$

$\frac{1}{2}\sin t\sin 2t = \sin^2 t\cos t$

$\frac{1}{4}\cos t\sin 2t = \frac{1}{2}\sin t\cos^2 t$

$$\underline{x} = c_1\begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2\begin{bmatrix} 5\sin t \\ 2\sin t - \cos t \end{bmatrix} - \begin{bmatrix} 5/2 \\ 1 \end{bmatrix} t\sin t + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} t\cos t$$

$$+ \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \sin t$$

$$= \underbrace{\left(c_1 + \tfrac{1}{2}\right)}_{d_1}\begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2\begin{bmatrix} 5\sin t \\ 2\sin t - \cos t \end{bmatrix} - \begin{bmatrix} 5/2 \\ 1 \end{bmatrix} t\sin t + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} t\cos t$$

$$- \begin{bmatrix} 5/2 \\ 1 \end{bmatrix} \cos t$$

7.9.12: $x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix}, \quad \pi/2 < t < \pi$

$|A - rI| = (2-r)(-2-r) + 5 = r^2 + 1 = 0, \quad r = \pm i$

$r_1 = i: \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow u = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$

$x_1 = (\cos t + i\sin t)\begin{pmatrix} 5 \\ 2-i \end{pmatrix} = \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + i\begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$

$x_g = c_1\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2\begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$

$\Psi = \begin{bmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & 2\sin t - \cos t \end{bmatrix} \quad |\Psi| = -5 \quad \Psi^{-1} = \begin{bmatrix} -\frac{2}{5}\sin t + \frac{1}{5}\cos t & \sin t \\ \frac{2}{5}\cos t + \frac{1}{5}\sin t & -\cos t \end{bmatrix}$

$u' = \Psi^{-1} g = \begin{bmatrix} -\frac{2}{5}\sin t + \frac{1}{5}\cos t & \sin t \\ \frac{2}{5}\cos t + \frac{1}{5}\sin t & -\cos t \end{bmatrix}\begin{bmatrix} \frac{1}{\sin t} \\ \frac{1}{\cos t} \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} + \frac{1}{5}\cot t + \tan t \\ \frac{2}{5}\cot t + \frac{1}{5} - 1 \end{bmatrix}$

$u = \int_{t_0}^{t} \Psi^{-1} g \, ds = \begin{bmatrix} -\frac{2}{5}t + \frac{1}{5}\ln(\sin t) - \ln(\cos t) + c_1 \\ \frac{2}{5}\ln(\sin t) - \frac{4}{5}t + c_2 \end{bmatrix}$

$x = \Psi u = \begin{bmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & 2\sin t - \cos t \end{bmatrix}\begin{bmatrix} -\frac{2}{5}t + \frac{1}{5}\ln(\sin t) - \ln(\cos t) + c_1 \\ \frac{2}{5}\ln(\sin t) - \frac{4}{5}t + c_2 \end{bmatrix}$

$$= \begin{bmatrix} -2t\cos t + \cos t\ln(\sin t) - 5\cos t\ln(\cos t) + 5c_1\cos t + 2\sin t\ln(\sin t) \\ -\frac{4}{5}t\cos t - \frac{2}{5}t\sin t + \frac{2}{5}\cos t\ln(\sin t) + \frac{1}{5}\sin t\ln(\sin t) - 2\cos t\ln(\cos t) \end{bmatrix}$$

$$+ \begin{bmatrix} -4t\sin t + 5c_2\sin t \\ -\sin t\ln(\cos t) + 2c_1\cos t + c_1\sin t + \frac{4}{5}\sin t\ln(\sin t) - \frac{2}{5}\cos t\ln\sin t \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ -\frac{8}{5}t\sin t + \frac{4}{5}t\cos t + 2c_2\sin t - c_2\cos t \end{bmatrix}$$

7.9.12:

$$x(t) = \begin{bmatrix} 5\cos t\left(-\frac{2}{5}t + \frac{1}{5}\ln(\sin t) - \ln(-\cos t) + c_1\right) \\ \left(-\frac{2}{5}t + \frac{1}{5}\ln(\sin t) - \ln(-\cos t) + c_1\right)(2\cos t + \sin t) \end{bmatrix}$$

$$+ \begin{bmatrix} 5\sin t\left(\frac{2}{5}\ln(\sin t) - \frac{4}{5}t + c_2\right) \\ \left(\frac{2}{5}\ln(\sin t) - \frac{4}{5}t + c_2\right)(-\cos t + 2\sin t) \end{bmatrix}$$

$$= \begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix}\left(-\frac{2}{5}t + \frac{1}{5}\ln(\sin t) - \ln(-\cos t) + c_1\right)$$

$$+ \begin{bmatrix} 5\sin t \\ -\cos t + 2\sin t \end{bmatrix}\left(-\frac{4}{5}t + \frac{2}{5}\ln(\sin t) + c_2\right)$$