7.6.1: $x' = \begin{pmatrix}3 & -2\\ 4 & -1\end{pmatrix}x$ $\quad |A-rI| = (3-r)(-1-r)+8$

$= r^2-2r+5=0 \quad r=\frac{1}{2}(2\pm\sqrt{4-20}) = \frac{1}{2}(2\pm 4i) = 1\pm 2i$

$r_1=1+2i: \begin{pmatrix}2-2i & -2\\ 4 & -2-2i\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \underline{u} = \begin{pmatrix}1+i\\ 2\end{pmatrix}$

$\underline{x} = c_1 e^{t} \begin{pmatrix}\cos 2t - \sin 2t\\ 2\cos 2t\end{pmatrix} + c_2 e^{t}\begin{pmatrix}\cos 2t+\sin 2t\\ 2\sin 2t\end{pmatrix}$

$= (c_1+c_2)e^{t}\begin{pmatrix}\cos 2t\\ 0\end{pmatrix} + (c_2-c_1)e^{t}\begin{pmatrix}\sin 2t\\ 0\end{pmatrix}$

$+ (c_1+c_2)e^{t}\begin{pmatrix}0\\ \cos 2t+\sin 2t\end{pmatrix} + (c_2-c_1)\begin{pmatrix}0\\ \sin 2t-\cos 2t\end{pmatrix}$

$= d_1 e^{t}\begin{pmatrix}\cos 2t\\ \cos 2t+\sin 2t\end{pmatrix} + d_2 e^{t}\begin{pmatrix}\sin 2t\\ -\cos 2t+\sin 2t\end{pmatrix}$

Solutions spiral outward counterclockwise to ∞

7.6.2: $x' = \begin{pmatrix}-1 & -4\\ 1 & -1\end{pmatrix}x$

$0=|A-rI| = (-1-r)(-1-r)+4 = r^2+2r+5 \Rightarrow r = \frac{-2\pm\sqrt{4-20}}{2}$

$r_1=-1+2i: \begin{pmatrix}-2i & 4\\ 1 & -2i\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow \underline{u} = \begin{pmatrix}2i\\ 1\end{pmatrix}$

$x_1 = e^{-t}\begin{pmatrix}2i\\ 1\end{pmatrix}(\cos(2t)+i\sin(2t))$

$\Rightarrow \underline{x} = c_1 e^{-t}\begin{pmatrix}2\cos 2t\\ \sin 2t\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}-2\sin(2t)\\ \cos 2t\end{pmatrix}$

Solutions spiral inward counterclockwise to 0.

7.6.3: $x'=\begin{pmatrix}2 & -5\\ 1 & -2\end{pmatrix}x$ $\quad |A-rI|=(2-r)(-2-r)+5$

$=r^2+1=0\Rightarrow r=\pm i$

$r_1=i: \begin{pmatrix}2-i & 5\\ 1 & -2-i\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow \underline{w}=\begin{pmatrix}2+i\\ 1\end{pmatrix}=\begin{pmatrix}5\\ 2-i\end{pmatrix}$

$x_1=e^{0}\begin{pmatrix}5\\ 2-i\end{pmatrix}(\cos t+i\sin t)$

$\Rightarrow \underline{x} = c_1\begin{pmatrix}5\cos t\\ 2\cos t+\sin t\end{pmatrix}+c_2\begin{pmatrix}5\sin t\\ -\cos t+2\sin t\end{pmatrix}$

Solutions oscillate around the origin counterclockwise, in an oval.

7.6.4: $x' = \begin{pmatrix}2 & -5/2\\ 9/5 & -1\end{pmatrix}x$ $\quad |A-rI| = (2-r)(-1-r)+\frac{9}{2}$

$=r^2-r+5/2=0\Rightarrow r=\frac{1\pm\sqrt{1-10}}{2}=\frac{1\pm 3i}{2}$

$r_1=\frac{1}{2}+\frac{3}{2}i; \begin{pmatrix}3/2-3/2i & -5/2\\ 9/5 & -3/2-3/2i\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow \underline{w}=\begin{pmatrix}5\\ 3-3i\end{pmatrix}$

$x_1=e^{\frac{1}{2}t}\begin{pmatrix}5\\ 3-3i\end{pmatrix}(\cos(\tfrac{3}{2}t)+i\sin(\tfrac{3}{2}t))$

$\Rightarrow \underline{x} = c_1 e^{\frac{1}{2}t}\begin{pmatrix}5\cos(\frac{3}{2}t)\\ 3\cos(\frac{3}{2}t)+3\sin(\frac{3}{2}t)\end{pmatrix}+c_2 e^{\frac{1}{2}t}\begin{pmatrix}5\sin(\frac{3}{2}t)\\ -3\cos(\frac{3}{2}t)+3\sin(\frac{3}{2}t)\end{pmatrix}$

Solutions spiral outward counterclockwise

7.6.5: $x' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x \quad |A - rI| = (1-r)(-3-r) + 5 = 0$
$= r^2 + 2r + 5 \Rightarrow r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$
$r_1 = -1 + 2i: \begin{pmatrix} 2-2i & -1 \\ 5 & -2-2i \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{\omega} = \begin{pmatrix} 1 \\ 2-2i \end{pmatrix}$
$\underline{x}_1 = e^{-t}\begin{pmatrix} 1 \\ 2-2i \end{pmatrix}(\cos(2t) + i\sin(2t))$
$\underline{x} = c_1 e^{-t}\begin{pmatrix} \cos(2t) \\ 2\cos(2t) + 2\sin(2t) \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin(2t) \\ -2\cos(2t) + 2\sin(2t) \end{pmatrix}$
Solutions spiral inward counterclockwise to (0,0).

7.6.6: $x' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} x \quad |A - rI| = (1-r)(-1-r) + 10$
$= r^2 + 9 = 0 \Rightarrow r = \pm 3i$
$r_1 = 3i: \begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{\omega} = \begin{pmatrix} 2 \\ 3i-1 \end{pmatrix}$
$\underline{x}_1 = \begin{pmatrix} 2 \\ 3i-1 \end{pmatrix}(\cos(3t) + i\sin(3t))$
$\underline{x} = c_1\begin{pmatrix} 2\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{pmatrix} + c_2\begin{pmatrix} 2\sin(3t) \\ 3\cos(3t) - \sin(3t) \end{pmatrix}$
Solutions oscillate around zero, in a left tilted oval

7.6.7: $x' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} x \quad |A - rI| = (1-r)((1-r)^2 + 4)$
$= (1-r)(r^2 - 2r + 5) = 0$
$\Rightarrow r_1 = 1,\ r_2 = 1+2i,\ r_3 = 1-2i$
$r_1: \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{\omega} = \begin{pmatrix} 1 \\ -3/2 \\ 1 \end{pmatrix}$
$r_2: \begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{z} = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$
$\underline{x}_2 = e^t\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}(\cos(2t) + i\sin(2t))$
$\underline{x} = c_1 e^t\begin{pmatrix} 1 \\ -3/2 \\ 1 \end{pmatrix} + c_2 e^t\begin{pmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{pmatrix} + c_3 e^t\begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix}$

The solution goes to ∞, oscillating in a cyllinder towards ∞.

7.6.8: $x' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} x \quad |A - rI| = (-3-r)(-1-r)(-r) + 2(-1-2-2r)$

$= (3+r)(1+r)(-r) + 2(-3-2r) = -r^3 - 4r^2 - 7r - 6$

$= -(r+2)(r^2+2r+3) = 0 \Rightarrow r_1 = -2,\ r_2 = -1+\sqrt{2}i,\ r_3 = -1-\sqrt{2}i$

$r_1:\ \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{w} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$r_2:\ \begin{pmatrix} -2+\sqrt{2}i & 0 & 2 \\ 1 & \sqrt{2}i & 0 \\ -2 & -1 & 1+\sqrt{2}i \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{z} = \begin{pmatrix} 2 \\ \sqrt{2}i \\ 2-\sqrt{2}i \end{pmatrix}$

$\underline{x}_2 = e^{-t}\begin{pmatrix} 2 \\ \sqrt{2}i \\ 2-\sqrt{2}i \end{pmatrix}\left(\cos(\sqrt{2}t) - i\sin(\sqrt{2}t)\right)$

$\underline{x} = c_1 e^{-2t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \\ 2\cos(\sqrt{2}t) + -\sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} -2\sin(\sqrt{2}t) \\ \sqrt{2}\cos(\sqrt{2}t) \\ -2\sin(\sqrt{2}t) - \sqrt{2}\cos(\sqrt{2}t) \end{pmatrix}$

The solution goes swiftly to the origin, oscillating in a plane.

7.6.9: $x' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} x,\ x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |A - rI| = (1-r)(-3-r) + 5$

$= r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$r_1 = -1+i:\ \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{w} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

$\underline{x}_1 = e^{-t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix}(\cos t + i\sin t)$

$x(t) = c_1 e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$

$x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow c_1 = 1,\ c_2 = -1$

$x(t) = e^{-t}\begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}$

The solution oscillates (spiraling inward) around 0, ccw.

7.6.10: $x' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} x,\ x(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad |A - rI| = (-3-r)(-1-r) + 2$

$= r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

$r_1 = -2+i:\ \begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{w} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

$\underline{x}_1 = e^{-2t}\begin{pmatrix} 1-i \\ 1 \end{pmatrix}(\cos t + i\sin t)$

$\underline{x}(t) = c_1 e^{-2t}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}$

$x(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow c_1 = -2,\ c_2 = -3$

$x(t) = e^{-2t}\begin{pmatrix} \cos t - 5\sin t \\ -2\cos t - 3\sin t \end{pmatrix}$

The solution spirals clockwise into the origin.

$7.6.21: \; t\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} = \xi t^r$

$|A - rI| = (-1-r)(-1-r) + 2 = r^2 + 2r + 3 = 0$

$\Rightarrow r = \frac{-2 \pm \sqrt{4-12}}{2} = -1 \pm \sqrt{2}\, i$

$r_1 = -1 + \sqrt{2} i: \; \begin{pmatrix} -\sqrt{2} i & -1 \\ 2 & -\sqrt{2} i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \xi = \begin{pmatrix} -1 \\ \sqrt{2} i \end{pmatrix}$

$\underline{x}_1 = \begin{pmatrix} -1 \\ \sqrt{2} i \end{pmatrix} t^{-1+\sqrt{2} i} = \begin{pmatrix} -1 \\ \sqrt{2} i \end{pmatrix} t^{-1} e^{\sqrt{2} i \ln t} = t^{-1} \begin{pmatrix} -1 \\ \sqrt{2} i \end{pmatrix} (\cos(\sqrt{2} \ln t) + i \sin(\sqrt{2} \ln t))$

$\underline{x} = c_1 t^{-1} \begin{pmatrix} -\cos(\sqrt{2} \ln t) \\ -\sqrt{2} \sin(\sqrt{2} \ln t) \end{pmatrix} + c_2 t^{-1} \begin{pmatrix} -\sin(\sqrt{2} \ln t) \\ \sqrt{2} \cos(\sqrt{2} \ln t) \end{pmatrix}$

$7.6.22: \; t\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} = \xi t^r$

$0 = |A - rI| = (2-r)(-2-r) + 5 = r^2 + 1 \Rightarrow r = \pm i$

$r_1 = i: \; \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \xi = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

$\underline{x}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} t^i = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{i \ln t} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} (\cos(\ln t) + i \sin(\ln t))$

$\underline{x} = c_1 \begin{pmatrix} 2\cos(\ln t) - \sin(\ln t) \\ \cos(\ln t) \end{pmatrix} + c_2 \begin{pmatrix} \cos(\ln t) + 2\sin(\ln t) \\ \sin(\ln t) \end{pmatrix}$

$7.6.26: \; \begin{pmatrix} I' \\ V' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}$

a) $|A - rI| = (-r)\left(\frac{-1}{RC} - r\right) + \frac{1}{LC} = r^2 + \frac{1}{RC} r + \frac{1}{LC} = 0$

$\Rightarrow r = \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{LC}}}{2} = \frac{1}{2}\left(\frac{-1}{RC} \pm \sqrt{\underbrace{\frac{L}{R^2C^2L} - \frac{4CR^2}{LC^2R^2}}}\right)$

If $L - 4CR^2 > 0$, the eigenvalues are clearly real and different, if $L - 4CR^2 < 0$, then we have a square root of a negative number, & the eigenvalues are complex conjugates.

7.6.26: b) $R=1, C=\frac{1}{2}, L=1$; $L-4R^2C=1-2=-1$

$\begin{pmatrix} I \\ V \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}$ $\quad |A-rI| = (-r)(-2-r)+2 =$

$r^2+2r+2 \Rightarrow r=\frac{-2\pm\sqrt{4-8}}{2} = -1\pm i$

$r_1=-1+i$: $\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{\xi} = \begin{pmatrix} 1 \\ i-1 \end{pmatrix}$

$\underline{x}_1 = e^{-t}\begin{pmatrix} 1 \\ i-1 \end{pmatrix}(\cos t + i\sin t)$

$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} \cos t \\ -\cos t-\sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ +\cos t-\sin t \end{pmatrix}$

c) $\begin{pmatrix} I \\ V \end{pmatrix}(0) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow c_1=2, c_2=3$

$I(t) = e^{-t}(2\cos t + 3\sin t)$

$V(t) = e^{-t}(\cos t - 5\sin t)$

d) $\lim(t\to\infty) I(t) = \lim_{t\to\infty} V(t) = 0$. These limits do not depend on initial conditions.

7.6.27: $r_1 = \lambda + i\mu$, $\bar{r}_1 = r_2 = \lambda - i\mu$, $\underline{\xi}' = \underline{a}+i\underline{b}$, $\bar{\underline{\xi}}' = \underline{a}-i\underline{b}$

a) $c_1\underline{a} + c_2\underline{b} = \underline{0}$; $\underline{a} = \frac{1}{2}\underline{\xi}' + \frac{1}{2}\bar{\underline{\xi}}'$, $\underline{b} = \frac{1}{2i}\underline{\xi}' - \frac{1}{2i}\bar{\underline{\xi}}'$

$\Rightarrow c_1\cdot\frac{1}{2}\underline{\xi}' + c_1\cdot\frac{1}{2}\bar{\underline{\xi}}' + c_2\cdot\frac{1}{2i}\underline{\xi}' - c_2\cdot\frac{1}{2i}\bar{\underline{\xi}}' = \underline{0}$

$\Rightarrow \underline{\xi}'\left(\frac{c_1}{2} + \frac{c_2}{2i}\right) + \bar{\underline{\xi}}'\left(\frac{c_1}{2} - \frac{c_2}{2i}\right) = \underline{0}$

multiply by 2, note $\frac{1}{i} = -i$

$\Rightarrow (c_1 - ic_2)\underline{\xi}' + (c_1+ic_2)\bar{\underline{\xi}}' = \underline{0}$

b) Since $\underline{\xi}'$ and $\bar{\underline{\xi}}'$ are linearly independent, it is clear that $(c_1-ic_2)=0$ and $(c_1+ic_2)=0$

Thus $c_1 = ic_2 \Rightarrow ic_2 + ic_2 = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 0$.

$\Rightarrow \underline{a}, \underline{b}$ are linearly independent

c) $c_1\underline{u}(t_0) + c_2\underline{v}(t_0) = \underline{0}$

$\Rightarrow c_1 e^{\lambda t_0}(\underline{a}\cos(\mu t_0) - \underline{b}\sin(\mu t_0)) + c_2 e^{\lambda t_0}(\underline{a}\sin(\mu t_0) + \underline{b}\cos(\mu t_0))$ $=0$. Since $e^{\lambda t_0} \neq 0$, we can divide it out, Thus

$$\Rightarrow \underline{a}\,\underbrace{(c_1\cos(\mu t_0) + c_2\sin(\mu t_0))}_{\alpha} + \underline{b}\,\underbrace{(-c_1\sin(\mu t_0) + c_2\cos(\mu t_0))}_{\beta} = 0$$

α & β are constants. Since $\underline{a}$ & $\underline{b}$ are linearly independent $\alpha = \beta = 0$

7.6.27: c) $\alpha = c_1 \cos(\mu t_0) + c_2 \sin(\mu t_0) = 0$

$\Rightarrow \quad c_1 \cos(\mu t_0) = -c_2 \sin(\mu t_0)$

$\sin(\mu t_0)$ and $\cos(\mu t_0)$ cannot be zero at the same time. Without loss of generality, we assume that $\sin(\mu t_0) \neq 0$.

Then $\quad c_2 = -c_1 \cot(\mu t_0)$

$\beta = 0 \Rightarrow \quad -c_1 \sin(\mu t_0) - c_1 \cot(\mu t_0) \cos \mu(t_0) = 0$

$\Rightarrow \quad -c_1 (\sin(\mu t_0) + \cot(\mu t_0)\cos(\mu t_0)) = 0$

$\Rightarrow \quad -c_1 (\sin^2(\mu t_0) + \cos^2(\mu t_0)) = 0$

$\Rightarrow \quad -c_1 = 0 \Rightarrow \quad c_2 = 0$

Since t_0 is arbitrary, this is try for all t_0.
Hence $\underline{u}(t)$ and $v(t)$ are linearly independent.

7.6.28: a) $mu'' + ku = 0$; $x_1 = u, x_2 = u'$

$\Rightarrow \quad m x_2' + k x_1 = 0 \quad , \quad x_1' = x_2 .$

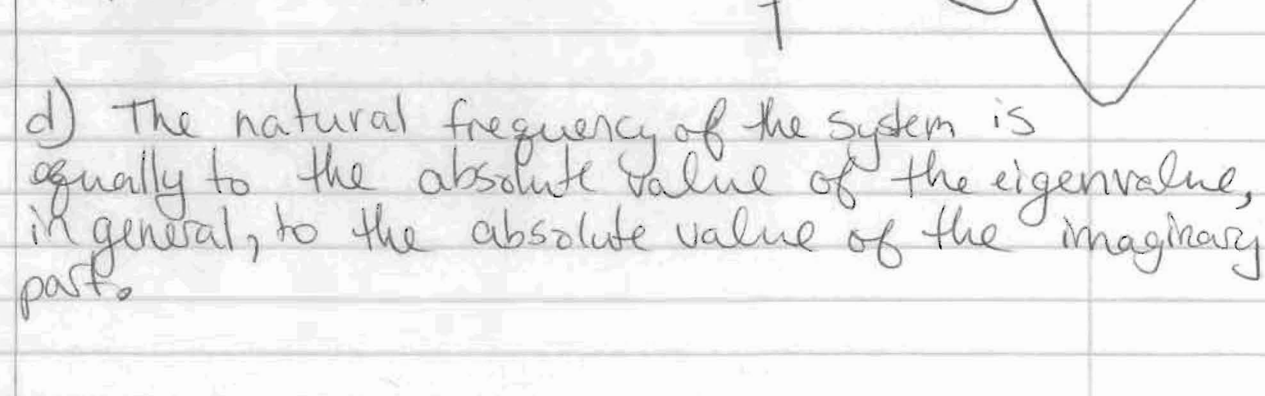
$\Rightarrow \quad x_2' = -\frac{k}{m} x_1 \quad , \quad x_1' = x_2$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

b) $|A - rI| = (-r)(-r) + k/m = r^2 + k/m \Rightarrow r = \pm\sqrt{-k/m}$

$\Rightarrow r = \pm i \sqrt{k/m}$

c) $\underline{X} = c_1 \begin{pmatrix} \cos(\sqrt{k/m}\, t) \\ -\sqrt{k/m} \sin(\sqrt{k/m}\, t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(\sqrt{k/m}\, t) \\ \sqrt{k/m} \cos(\sqrt{k/m}\, t) \end{pmatrix}$



d) The natural frequency of the system is equally to the absolute value of the eigenvalue, in general, to the absolute value of the imaginary part.