Discontinuous Galerkin methods
Lecture 7

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8.4 Scattering about a vertical cylinder in a finite-width channel

Figure 8.12: Scattering of waves about a cylinder in a finite-width channel.

A numerical study is carried out to both test the consistency of the imposed boundary conditions and investigate how the geometric representation of the domain may affect the computed solution. In a numerical model based on the linear Padé (2,2) rotational velocity version we set up a test for open-channel flow. We seek to model the scattering of an incident wave field which is propagating toward a bottom-mounted rigid cylinder positioned in the middle of a finite-width channel. Due to the symmetry, the solution is mathematically equivalent to the solution of an infinite row of cylinders positioned perpendicular to the wave propagation direction. For example, see Linton and Evans (1993) and Linton (2005). The domain can be represented by a channel with a cylinder in the middle or alternatively on a half-sized domain with rigid walls at the symmetry lines of the solution. The latter approach reduces the domain to the smallest size, which is convenient computationally and this approach is therefore used.

To consider how the geometric representation of the domain may affect the resulting wave field and the maximum wave run up at the cylinder surface boundary, we consider three different unstructured grids. The grids shown in Figures 8.13 a)-b) have comparable spatial resolution away from the cylinder surface, such that the significant differences between the grid lies in the spatial resolution (measured by the size and number of elements) in the immediate neighborhood of the cylinder surface and the representation (or rather approximation) of the circular cylinder surface. Thus the grid in Figure 8.13 a) is defined using curvilinear elements for the accurate representation of the cylinder boundary (thus the representation will no longer be polygonal but curved at the boundary). For the straight-sides representation of the cylinder surface we use both the coarse grid in 8.13 a) and the locally refined grid in 8.13 b).

In each case, a combined wave generation and wave absorption zone is introduced in the region $-5 \leq x/L \leq -3.5$ in the western end of the channel. A sponge layer is positioned at the eastern end in the region $3 \leq x/L \leq 5$. The incident wave field consists of plane waves propagating parallel to the channel walls. The wave length is set to $L = 1$ m, the cylinder radius is a quarter of the width of the channel, $a = 0.5$ m, and hence the dimensionless radius is $ka = \pi$. The angular wave frequency $\omega$ is determined from the linear Boussinesq dispersion relation given in Eq. (2.66). In all tests the time increment is chosen...
A brief overview of what’s to come

• Lecture 1: Introduction, Motivation, History
• Lecture 2: Basic elements of DG-FEM
• Lecture 3: Linear systems and some theory
• Lecture 4: A bit more theory and discrete stability
• Lecture 5: Attention to implementations
• Lecture 6: Nonlinear problems and properties
• Lecture 7: Problems with discontinuities and shocks
• Lecture 8: Higher order/Global problems
Lecture 7

- Let’s summarize the smooth case
- Gibbs Phenomenon
- Filtering and post-processing
- Limiting
- TVD Runge-Kutta Methods
- A few theoretical results
- Solving the Euler equations
Lets summarize

We have achieved a good understanding

• The theoretical support for DG for conservation laws is very solid.
• The requirements for ‘exact’ integration is expensive.
• It seems advantageous to consider a nodal approach in combination with dissipation.
• Dissipation can be implemented using a filter
• There is a complete error-theory for smooth problems.

... but we have ‘forgotten’ the unpleasant issue

What about discontinuous solutions?
Let us first consider a simple approximation

\[ u(x) = -\text{sign}(x), \quad x \in [-1, 1], \]

- Overshoot does not go away with \( N \)
- First order point wise accuracy
- Oscillations are global

Gibbs Phenomenon
Gibbs Phenomenon

But do the oscillations destroy the nice behavior?

\[ \frac{\partial u}{\partial t} + a(x) \frac{\partial u}{\partial x} = 0, \quad u(t, 0, x) = f(x), \]

\[ \frac{\partial v}{\partial t} - \mathcal{L}^* v = 0, \]

\( a(x) \) is smooth - but \( u(x, 0) \) is not

Define the adjoint problem

Clearly, we have

\[ \frac{d}{dt} (u, v)_{\Omega} = 0 \Rightarrow (u(t), v(t))_{\Omega} = (u(0), v(0))_{\Omega}. \]
Using central fluxes, we also have

\[(u_h(t), v_h(t))_{\Omega,h} = (u_h(0), v_h(0))_{\Omega,h}.\]

Consider

\[(u_h(0), v_h(0))_{\Omega,h} = (u(0), v(0))_{\Omega} + (u_h(0) - u(0), v_h(0))_{\Omega,h} + (u(0), v_h(0) - v(0))_{\Omega,h}.\]

We also have

\[(u_h(0), v_h(0))_{\Omega,h} \leq (u(0), v(0))_{\Omega} + C(u)h^{N+1}N^{-q} |v(0)|_{\Omega,q}.\]

\[||v(t) - v_h(t)||_{\Omega,h} \leq C(t)\frac{h^{N+1}}{Nq} |v(t)|_{\Omega,q};\]

Combining it all, we obtain

\[(u_h(t), v(t))_{\Omega,h} = (u(t), v(t))_{\Omega} + \varepsilon,\]
Gibbs Phenomenon

The solution is spectrally accurate! ... but it is ‘hidden’

This also shows that the high-order accuracy is maintained -- ‘the oscillations are not noise’!

How do we recover the accurate solution?

Recall

\[ u_h(x) = \sum_{n=1}^{N_P} \hat{u}_n \tilde{P}_{n-1}(x), \quad \hat{u}_n = \int_{-1}^{1} u(x) \tilde{P}_{n-1}(x) \, dx. \]

One easily shows that

\[ u(x) \in H^q \Rightarrow \hat{u}_n \propto n^{-q} \]
Filtering

So there is a close connection between smoothness and decay for the expansion coefficients.

Perhaps we can ‘convince’ the expansion do decay faster?

Consider

\[ u_h^F(x) = \sum_{n=1}^{N_p} \sigma \left( \frac{n-1}{N} \right) \hat{u}_n \tilde{P}_{n-1}(x). \]

\[ \sigma(\eta) = \exp(-\alpha \eta^s) \]

Example

\[ u^{(0)} = \begin{cases} 
-\cos(\pi x), & -1 \leq x \leq 0 \\
\cos(\pi x), & 0 < x \leq 1,
\end{cases} \]

\[ u^{(i)} = \int_{-1}^{x} u^{(i-1)}(s) \, ds, \]
Filtering

Fig. 5.5. In the left column is shown the pointwise error after the exponential filter with $s = 2$ has been applied to the Legendre expansions of the four test functions. We use $N = 16$, $N = 64$, and $N = 256$ for each function. The middle column shows similar results for $s = 6$ and the right column displays the results for $s = 10$.

For the $s = 2$ filter (first column in Fig. 5.5), which limits the convergence rate regardless of the regularity of the function being approximated. However, if $s$ is sufficiently large, this does not adversely impact the pointwise error when filtering smooth functions. The results in Fig. 5.5 highlight that the filter does not affect the smooth functions.
Filtering

This achieves exactly what we hoped for

• Improves the accuracy away from the problem spot
• Does not destroy matter at the problem spot
  ... but does not help there.

This suggests a strategy:

• Use a filter to stabilize the scheme but do not remove the oscillations.
• Postprocess the data after the end of the computation.
Filtering

Consider Burgers equation

\[ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} = 0, \ x \in [-1, 1], \]

\[ u_0(x) = u(x, 0) = \begin{cases} 2, & x \leq -0.5 \\ 1, & x > -0.5. \end{cases} \]

\[ u(x, t) = u_0(x - 3t), \]

Overfiltering leads to severe smearing.

Limited filtering looks much better
Filtering

An alternative - Pade filtering

\[ u^k_h(x) = \frac{R_M(x)}{Q_L(x)}, \]

To fully recover, the shock location is required (see text).

- Eliminates oscillations and improves accuracy
- .. but no improvement at the point
Limiting

So for some/many problems, we could simply leave the oscillations -- and then postprocess.

However, for some applications (.. and advisors) this is not acceptable

- Unphysical values (negative densities)
- Artificial events (think combustion)
- Visually displeasing (.. for the advisor).

So we are looking for a way to completely remove the oscillations:

Limiting
Limiting

We are interested in guaranteeing uniform boundedness

\[ \|u\|_{L^1} \leq C, \quad \|u\|_{L^1} = \int_{\Omega} |u| \, dx. \]

Consider

\[ \frac{\partial}{\partial t} u^\varepsilon + \frac{\partial}{\partial x} f(u^\varepsilon) = \varepsilon \frac{\partial^2}{\partial x^2} u^\varepsilon. \quad \text{and define} \quad \eta(u) = |u| \]

We have

\[ -\int_{\Omega} (\eta'(u_x)) u_t \, dx = \int_{\Omega} \frac{u_x}{|u_x|} u_{xt} \, dx = \frac{d}{dt} \int_{\Omega} |u_x| \, dx = \frac{d}{dt} \|u_x\|_{L^1}. \]

and one easily proves

\[ \frac{d}{dt} \|u_x^\varepsilon\|_{L^1} \leq 0. \]
Limiting

We would like to repeat this for the discrete scheme.

Consider first the N=0 FV scheme

\[ h \frac{d u_h^k}{d t} + f^*(u_h^k, u_h^{k+1}) - f^*(u_h^k, u_h^{k-1}) = 0, \]

Multiply with

\[ v_h^k = -\frac{1}{h} \left[ \eta' \left( \frac{u_h^{k+1} - u_h^k}{h} \right) - \eta' \left( \frac{u_h^k - u_h^{k-1}}{h} \right) \right] \]

and sum over all elements to get

\[ \frac{d}{d t} |u_h|_{TV} + \sum_{k=1}^{K} v_h^k \left( f^*(u_h^k, u_h^{k+1}) - f^*(u_h^k, u_h^{k-1}) \right) = 0, \]

\[ |u_h|_{TV} = \sum_{k=1}^{K} |u_h^{k+1} - u_h^k|. \]
Limiting

Using that the flux is monotone, one easily proves

\[ v^k_h \left( f^* (u^k_h, u^k_{h+1}) - f^* (u^k_h, u^k_{h-1}) \right) \geq 0 \]

and therefore

\[ \frac{d}{dt} |u_h|_{TV} \leq 0, \]

So for N=0 everything is fine -- but what about N>0

\[ h \frac{d\tilde{u}^k_h}{dt} + f^* (u^k_r, u^k_{l+1}) - f^* (u^k_l, u^k_{r-1}) = 0, \]

using a Forward Euler method in time, we get

\[ \frac{h}{\Delta t} \left( \tilde{u}^{k,n+1} - \tilde{u}^{k,n} \right) + f^* (u^k_r, u^k_{l+1,n}) - f^* (u^k_l, u^k_{r-1,n}) = 0, \]
Limiting

Resulting in

\[ |\bar{u}^{n+1}|_{TV} - |\bar{u}^n|_{TV} + \Phi = 0, \]

However, the monotone flux is not enough to guarantee uniform boundedness through \( \Phi \geq 0 \)

That is the job of the limiter -- which must satisfy

- Ensures uniform boundedness/control oscillations
- Does not violate conservation
- Does not change the formal/high-order accuracy

This turns out to be hard!
Limiting

Two tasks at hand

• Detect troubled cells
• Limit the slope to eliminate oscillations

Define the \textit{minmod} function

\[
m(a_1, \ldots, a_m) = \begin{cases} 
  s \min_{1 \leq i \leq m} |a_i|, & |s| = 1 \\
  0, & \text{otherwise,}
\end{cases}
\]

\[
s = \frac{1}{m} \sum_{i=1}^{m} \text{sign}(a_i).
\]

If \(a\) are slopes, the \textit{minmod} function
• Returns the minimum slope if all have the same sign
• Returns slope zero if the slopes are different
Limiting

Let us assume \( N=1 \) in which case the solution is

\[
u^k_h(x) = \bar{u}_h^k + (x - x_0^k)(u^k_h)_x,\]

We have the classic MUSCL limiter

\[
\Pi^1 u_h^k(x) = \bar{u}_h^k + (x - x_0^k)m \left( (u^k_h)_x, \frac{\bar{u}_h^{k+1} - \bar{u}_h^k}{h}, \frac{\bar{u}_h^k - \bar{u}_h^{k-1}}{h} \right),
\]

or a slightly less dissipative limiter

\[
\Pi^1 u_h^k(x) = \bar{u}_h^k + (x - x_0^k)m \left( (u^k_h)_x, \frac{\bar{u}_h^{k+1} - \bar{u}_h^k}{h/2}, \frac{\bar{u}_h^k - \bar{u}_h^{k-1}}{h/2} \right),
\]

There are many other types but they are similar
Limiting

Consider

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad x \in [-1, 1], \]

Smooth initial condition

Reduction to 1st order at local smooth extrema
Limiting

Introduce the TVB minmod

\[
\bar{m}(a_1, \ldots, a_m) = m(a_1, a_2 + Mh^2\text{sign}(a_2), \ldots, a_m + Mh^2\text{sign}(a_m)),
\]

M estimates maximum curvature
Consider Burgers' equation

\[
\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} = 0, \quad x \in [-1, 1],
\]

\[
u_0(x) = u(x, 0) = \begin{cases} 
2, & x \leq -0.5 \\
1, & x > -0.5.
\end{cases}
\]

\[u(x, t) = u_0(x - 3t),\]

Too dissipative limiting leads to severe smearing.

.. but no oscillations!
Limiting

But what about N>1?

- Compare limited and nonlimited interface values
- If equal, no limiting is needed.
- If different, reduce to N=1 and apply slope limiting
Limiting

General remarks on limiting

• The development of a limiting technique that avoid local reduction to 1st order accuracy is likely the most important outstanding problem in DG

• There are a number of techniques around but they all have some limitations -- restricted to simple/equidistant grids, not TVD/TVB etc

• The extensions to 2D/3D and general grids are very challenging
TVD Runge-Kutta methods

Consider again the semi-discrete scheme

\[ \frac{d}{dt} u_h = \mathcal{L}_h(u_h, t), \]

For which we just discussed TVD/TVB schemes as

\[ u^{n+1}_h = u^n_h + \Delta t \mathcal{L}_h(u^n_h, t^n), \quad |u^{n+1}_h|_{TV} \leq |u^n_h|_{TV}. \]

.. but this is just 1st order in time -- we want high-order accuracy

\[ \textbf{Do we have to redo it all?} \]
TVD Runge-Kutta methods

Assume we can find an ERK method on the form

\[
\begin{cases}
    v^{(0)} = u^n_h \\
    i = 1, \ldots, s : v^{(i)} = \sum_{j=0}^{i-1} \alpha_{ij} v^{(j)} + \beta_{ij} \Delta t L_h(v^{(j)}, t^n + \gamma_j \Delta t) \\
    u_{h}^{n+1} = v^{(s)}
\end{cases}
\]

Coefficients found to satisfy order conditions

Write this as

\[
v^{(i)} = \sum_{j=0}^{i-1} \alpha_{ij} \left( v^{(j)} + \frac{\beta_{ij}}{\alpha_{ij}} \Delta t L_h(v^{(j)}, t^n + \gamma_j \Delta t) \right).
\]

Clearly if \( \alpha_{ij}, \beta_{ij} > 0 \)
TVD Runge-Kutta methods

Assume we can find a ERK method on the form

\[
\begin{cases}
  v^{(0)} = u_h^n \\
  i = 1, \ldots, s : v^{(i)} = \sum_{j=0}^{i-1} \alpha_{ij} v^{(j)} + \beta_{ij} \Delta t \mathcal{L}_h(v^{(j)}, t^n + \gamma_j \Delta t) \\
  u_{h}^{n+1} = v^{(s)}
\end{cases}
\]

Coefficients found to satisfy order conditions

Write this as

\[
v^{(i)} = \sum_{j=0}^{i-1} \alpha_{ij} \left( v^{(j)} + \frac{\beta_{ij}}{\alpha_{ij}} \Delta t \mathcal{L}_h(v^{(j)}, t^n + \gamma_j \Delta t) \right).
\]

Clearly if \( \alpha_{ij}, \beta_{ij} > 0 \)

The scheme is a convex combination of Euler steps and the stability of the high-order methods follows
TVD Runge-Kutta methods

... but do such schemes exits?

2nd order

\[ v^{(1)} = u^n_h + \Delta t L_h(u^n_h, t^n), \]
\[ u^{n+1}_h = v^{(2)} = \frac{1}{2} \left( u^n_h + v^{(1)} + \Delta t L_h(v^{(1)}, t^n + \Delta t) \right), \]

3rd order

\[ v^{(1)} = u^n_h + \Delta t L_h(u^n_h, t^n), \]
\[ v^{(2)} = \frac{1}{4} \left( 3u^n_h + v^{(1)} + \Delta t L_h(v^{(1)}, t^n + \Delta t) \right), \]
\[ u^{n+1}_h = v^{(3)} = \frac{1}{3} \left( u^n_h + 2v^{(2)} + 2\Delta t L_h \left( v^{(2)}, t^n + \frac{1}{2}\Delta t \right) \right). \]

No 4th order, 4 stage scheme is possible - but there are other options (not implicit)

With filter/limiting

\[ v^{(i)} = \Pi^p \left( \sum_{l=0}^{i-1} \alpha_{il} v^{(l)} + \beta_{il} \Delta t L_h(v^{(l)}, t^n + \gamma_l \Delta t) \right). \]
Example

\[ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} = 0, \quad x \in [-1, 1], \]

\[ u_0(x) = u(x, 0) = \begin{cases} 
2, & x \leq -0.5 \\
1, & x > -0.5. 
\end{cases} \]

\[ u(x, t) = u_0(x - 3t), \]

Use ‘standard’ 2nd order ERK

\[ v^{(1)} = u^n_h - 20 \Delta \mathcal{L}_h(u^n_h), \]

\[ u^{n+1}_h = u^n_h + \frac{\Delta t}{40} \left( 41 \mathcal{L}_h(u^n_h) - \mathcal{L}_h(v^{(1)}) \right). \]

Compare to 2nd order TVD-RK

MUSCL limiting in space, i.e., no oscillations
TVD Runge-Kutta methods

The oscillation is caused by time-stepping!

The 2nd order ERK is a bit unusual and ‘reasonable’ ERK method typically do not show this.

However, only with TVD-RK can one guarantee it
A few theoretical results

Theorem 5.12. Assume that the limiter, $\Pi$, ensures the TVDM property; that is,

$$v_h = \Pi(u_h) \Rightarrow |v_h|_{TV} \leq |u_h|_{TV},$$

and that the SSP-RK method is consistent.

Then the DG-FEM with the SSP-RK solution is TVDM as

$$\forall n: |u_h^n|_{TV} \leq |u_h^0|_{TV}.$$
A few theoretical results

**Theorem 5.12.** Assume that the limiter, $\Pi$, ensures the TVDM property; that is,

$$v_h = \Pi(u_h) \Rightarrow |v_h|_{TV} \leq |u_h|_{TV},$$

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Then the DG-FEM with the SSP-RK solution is TVDM as

$$\forall n: |u^n_h|_{TV} \leq |u^0_h|_{TV}.$$

**Theorem 5.14.** Assume that the slope limiter, $\Pi$, ensures that $u_h$ is TVDM or TVBM and that the SSP-RK method is consistent.

Then there is a subsequence, $\{\bar{u}_h\}'$, of the sequence $\{\bar{u}_h\}$ generated by the scheme that converges in $L^\infty(0,T;L^1)$ to a weak solution of the scalar conservation law.

Moreover, if a TVBM limiter is used, the weak solution is the entropy solution and the whole sequence converges.

Finally, if the generalized slope limiter guarantees that

$$\|\bar{u}_h - \Pi \bar{u}_h\|_{L^1} \leq Ch|\bar{u}_h|_{TV},$$

then the above results hold not only for the sequence of cell averages, $\{\bar{u}_h\}$, but also for the sequence of functions, $\{u_h\}$. 

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**Fig. 5.13.** Solution of Burgers’ equation with a shock using limiting. We use $K = 100$ and $N = 1$ order elements and the MUSCL slope limiter. The results are shown at $T = 0$.

4. On the left is shown the result computed using a generic second-order RK method and the figure on the right shows the nonoscillatory results obtained by the use of the SSP-RK2 method.

The results in Fig. 5.13 highlight the possibility of generating spurious oscillations in the solution solely by failing to use a suitable timestepping method.

5.8 A few general results

With all the pieces in place, one can establish a number of more general results regarding convergence for nonlinear scalar conservation laws with convex fluxes. For completeness, we will summarize a few of them here without proofs.

**Theorem 5.12.** Assume that the limiter, $\Pi$, ensures the TVDM property; that is,

$$v_h = \Pi(u_h) \Rightarrow |v_h|_{TV} \leq |u_h|_{TV},$$

and that the SSP-RK method is consistent.

Then the DG-FEM with the SSP-RK solution is TVDM as

$$\forall n: |u^n_h|_{TV} \leq |u^0_h|_{TV}.$$

This is an almost immediate consequence of the results we have discussed above.

Furthermore, we have similar results as follows:

**Theorem 5.13.** Assume that the limiter, $\Pi$, ensures the TVBM property and that the SSP-RK method is consistent.

Then the DG-FEM with the SSP-RK solution is TVBM as

$$\forall n: |u^n_h|_{TV} \leq |u^0_h|_{TV} + CM,$$

where the constant, $C$, depends only on the order of approximation, $N$, and $M$ is the constant in the TVBM limiter, Eq. (5.29).

These results allow one to prove the fundamental convergence theorem [60]

**Theorem 5.14.** Assume that the slope limiter, $\Pi$, ensures that $u_h$ is TVDM or TVBM and that the SSP-RK method is consistent.

Then there is a subsequence, $\{\bar{u}_h\}'$, of the sequence $\{\bar{u}_h\}$ generated by the scheme that converges in $L^\infty(0,T;L^1)$ to a weak solution of the scalar conservation law.

Moreover, if a TVBM limiter is used, the weak solution is the entropy solution and the whole sequence converges.

Finally, if the generalized slope limiter guarantees that

$$\|\bar{u}_h - \Pi \bar{u}_h\|_{L^1} \leq Ch|\bar{u}_h|_{TV},$$

then the above results hold not only for the sequence of cell averages, $\{\bar{u}_h\}$, but also for the sequence of functions, $\{u_h\}$. 

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5.9 The Euler equations of compressible gas dynamics

To conclude this chapter, let us consider a more elaborate example to see how all the pieces come together. We consider the one-dimensional equations of gas dynamics, known as the Euler equations. These are a set of three coupled nonlinear conservation laws given as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0,$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} = 0,$$

$$\frac{\partial E}{\partial t} + \frac{\partial (E + p) u}{\partial x} = 0,$$

where we have the conserved variables of density, $\rho$, momentum, $\rho u$, and energy, $E$. The energy and the pressure are related through the ideal gas law as
Solving the Euler equations

\[
\begin{align*}
&\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \quad \text{Mass} \\
&\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} = 0, \quad \text{Momentum} \\
&\frac{\partial E}{\partial t} + \frac{\partial (E + p)u}{\partial x} = 0, \quad \text{Energy}
\end{align*}
\]

\[p = (\gamma - 1) \left( E - \frac{1}{2} \rho u^2 \right), \quad c = \sqrt{\frac{\gamma p}{\rho}}, \quad \text{Ideal gas}\]

Sod’s Problem

\[
\begin{align*}
\rho(x, 0) &= \begin{cases} 1.0, & x < 0.5 \\ 0.125, & x \geq 0.5 \end{cases}, \quad \rho u(x, 0) = 0 \\
E(x, 0) &= \frac{1}{\gamma - 1} \begin{cases} 1, & x < 0.5 \\ 0.1, & x \geq 0.5. \end{cases}
\end{align*}
\]
Solving the Euler equations

Fig. 5.14. Solution of Sod’s shock tube problem at $T=0$ with $K=250$ linear elements and the MUSCL TVBM limiter. Shown is the computed density ($\rho$), the velocity ($u$), the pressure ($p$), and the local Mach number ($M$). The dashed lines represent the exact solution.

where $\xi(x, t) = x + 2\sqrt{GHt}$

$H$ is the steady-state water height and $G$ is the constant of gravity.

2. Consider the linear wave problem

$$\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = 0, \quad x \in [-1, 1],$$

$K=250, N=1, MUSCL$
Solving the Euler equations

Fig. 5.15. Solution of Sod’s shock tube problem at $T=0.2$ with $K=500$ linear elements and the MUSCL TVBM limiter. Shown is the computed density ($\rho$), the velocity ($u$), the pressure ($p$), and the local Mach number ($M$). The dashed lines represent the exact solution.

where $a(x) = \frac{1}{\pi} \sin(\pi x - 1)$.

If the initial condition is $u(x, 0) = f(x) = \sin(x)$, the exact solution to the linear problem is

$$u(x, t) = f(2 \tan^{-1}\left[\frac{e^{-t \tan(\pi x - 1)} - 1}{e^{-t \tan(\pi x - 1)} + 1}\right]) + 1.$$

a) Confirm that $u(x, t)$ is an exact solution and that the solution asymptotes to a constant. Find the constant.

b) Formulate a DG-FEM method with exact integration for solving the problem and implement the scheme. Use the exact solution as the boundary condition at $x = -1$ and confirm the accuracy of the scheme. Is it as expected?
Summary

Dealing with discontinuous problems is a challenge

• The Gibbs oscillations impact accuracy
• .. but it does not destroy it, it seems
• So they should not just be removed
• One can the try to postprocess by filtering or other techniques.
• For some problems, true limiting is required
• Doing this right is complicated -- and open
• TVD-RK allows one to prove nonlinear results
• .. and it all works :-)

Time to move beyond the 1st order problem