8.4 Scattering about a vertical cylinder in a finite-width channel

A numerical study is carried out to both test the consistency of the imposed boundary conditions and investigate how the geometric representation of the domain may affect the computed solution. In a numerical model based on the linear Padé (2,2) rotational velocity version we set up a test for open-channel flow. We seek to model the scattering of an incident wave field which is propagating toward a bottom-mounted rigid cylinder positioned in the middle of a finite-width channel.

Due to the symmetry, the solution is mathematically equivalent to the solution of an infinite row of cylinders positioned perpendicular to the wave propagation direction. For example, see Linton and Evans (1993) and Linton (2005). The domain can be represented by a channel with a cylinder in the middle or alternatively on a half-sized domain with rigid walls at the symmetry lines of the solution. The latter approach reduces the domain to the smallest size, which is convenient computationally and this approach is therefore used.

To consider how the geometric representation of the domain may affect the resulting wave field and the maximum wave run up at the cylinder surface boundary, we consider three different unstructured grids. The grids shown in Figures 8.13 a)-b) have comparable spatial resolution away from the cylinder surface, such that the significant differences between the grid lies in the spatial resolution (measured by the size and number of elements) in the immediate neighborhood of the cylinder surface and the representation (or rather approximation) of the circular cylinder surface. Thus the grid in Figure 8.13 a) is defined using curvilinear elements for the accurate representation of the cylinder boundary (thus the representation will no longer be polygonal but curved at the boundary). For the straight-sides representation of the cylinder surface we use both the coarse grid in 8.13 a) and the locally refined grid in 8.13 b).

In each case, a combined wave generation and wave absorption zone is introduced in the region $-5 \leq x/L \leq -3.5$ in the western end of the channel. A sponge layer is positioned at the eastern end in the region $3.0 \leq x/L \leq 5.0$. The incident wave field consists of plane waves propagating parallel to the channel walls. The wavelength is set to $L = 1$ m, the cylinder radius is a quarter of the width of the channel, $a = 0.5$ m, and hence the dimensionless radius is $ka = \pi$. The angular wave frequency $\omega$ is determined from the linear Boussinesq dispersion relation given in Eq. (2.66). In all tests the time increment is chosen...
A brief overview of what’s to come

• Lecture 1: Introduction, Motivation, History
• Lecture 2: Basic elements of DG-FEM
• Lecture 3: Linear systems and some theory
• Lecture 4: A bit more theory and discrete stability
• Lecture 5: Attention to implementations
• Lecture 6: Nonlinear problems and properties
• Lecture 7: Problems with discontinuities and shocks
• Lecture 8: Higher order/Global problems
Lecture 3

• Let’s briefly recall what we know
• Constructing fluxes for linear systems
• The local basis and Legendre polynomials
• Approximation theory on the interval
Let us recall

We already know a lot about the basic DG-FEM

- **Stability** is provided by carefully choosing the numerical flux.
- **Accuracy** appear to be given by the local solution representation.
- We can utilize major advances on monotone schemes to design fluxes.
- The scheme generalizes with very few changes to very general problems -- multidimensional systems of conservation laws.
Let us recall

We already know a lot about the basic DG-FEM

• **Stability** is provided by carefully choosing the numerical flux.
• **Accuracy** appear to be given by the local solution representation.
• We can utilize major advances on monotone schemes to design fluxes.
• The scheme generalizes with very few changes to very general problems -- multidimensional systems of conservation laws.

At least in principle -- but what can we actually prove?
But first a bit more on fluxes

Let us briefly look a little more carefully at linear systems

\[ Q(x) \frac{\partial u}{\partial t} + \nabla \cdot \mathcal{F} = Q(x) \frac{\partial u}{\partial t} + \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0, \]

\[ \mathcal{F} = [F_1, F_2] = [A_1(x)u, A_2(x)u]. \]

Prominent examples are

- Acoustics
- Electromagnetics
- Elasticity

In such cases we can derive exact upwind fluxes
Linear systems and fluxes

Assume first that all coefficients vary smoothly

\[ Q(x) \frac{\partial u}{\partial t} + A_1(x) \frac{\partial u}{\partial x} + A_2(x) \frac{\partial u}{\partial y} + B(x)u = 0, \]

The flux along a normal \( \hat{n} \) is then

\[ \Pi = (\hat{n}_x A_1(x) + \hat{n}_y A_2(x)). \quad \hat{n} \cdot \mathcal{F} = \Pi u. \]

Now diagonalize this as

\[ Q^{-1} \Pi = SA \Lambda S^{-1}, \]

\[ \Lambda = \Lambda^+ + \Lambda^-, \]

and we obtain

\[ (\hat{n} \cdot \mathcal{F})^* = QS (\Lambda^+ S^{-1} u^- + \Lambda^- S^{-1} u^+), \]
Linear systems and fluxes

For non-smooth coefficients, it is a little more complex

Consider the problem

\[
\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} = 0, \quad x \in [a, b].
\]

Then we clearly have

\[
\frac{d}{dt} \int_a^b u \, dx = -\lambda (u(b, t) - u(a, t)) = f(a, t) - f(b, t),
\]

\[
\frac{d}{dt} \int_a^b u \, dx = \frac{d}{dt} ((\lambda t - a)u^- + (b - \lambda t)u^+) = \lambda (u^- - u^+).
\]
Linear systems and fluxes

Hence, by simple mass conservation, we achieve

\[-\lambda(u^- - u^+) + (f^- - f^+) = 0.\]

for \(a \to x^-, b \to x^+\)

These are the Rankine-Hugoniot conditions

For the general system, these are

\[\forall i: -\lambda_i Q[u^- - u^+] + [(IIu)^- - (IIu)^+] = 0,\]

They must hold across each wave and can be used to connect across the interface.
Linear systems and fluxes

So for the 3-wave problem we have

\[ \lambda Q^- (u^* - u^-) + [(\Pi u)^* - (\Pi u)^-] = 0, \]
\[ [(\Pi u)^* - (\Pi u)^{**}] = 0, \]
\[ -\lambda Q^+ (u^{**} - u^+) + [(\Pi u)^{**} - (\Pi u)^+] = 0, \]

and the numerical flux is given as

\[ (\hat{n} \cdot \mathbf{F})^* = (\Pi u)^* = (\Pi u)^{**}, \]

This approach is general and yields the exact upwind fluxes -- but requires that the system can be solved !
Linear systems and fluxes -- an example

Consider
\[
\frac{\partial q}{\partial t} + A \frac{\partial q}{\partial x} = \frac{\partial}{\partial t} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} a(x) & 0 \\ 0 & -a(x) \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \end{bmatrix} = 0,
\]

Following the general approach, we have
\[
a^- (q^* - q^-) + (\Pi q)^* - (\Pi q)^- = 0,
- a^+ (q^* - q^+) + (\Pi q)^* - (\Pi q)^+ = 0,
\]

with \((\Pi q)^\pm = \hat{n} \cdot (Aq)^\pm = \hat{n} \cdot \begin{bmatrix} a^\pm & 0 \\ 0 & -a^\pm \end{bmatrix} \begin{bmatrix} u^\pm \\ v^\pm \end{bmatrix} = \hat{n} \cdot \begin{bmatrix} a^\pm u^\pm \\ -a^\pm v^\pm \end{bmatrix}.
\]

Solving this yields
\[
(\Pi q)^* = \frac{2a^+ a^-}{a^+ + a^-} \hat{n} \cdot \left( \begin{bmatrix} \{u\} \\ -\{v\} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} [u] \\ [v] \end{bmatrix} \right),
\]

Intermediate velocity
\[
a^* = \frac{2a^- a^+}{a^+ + a^-},
\]
Linear systems and fluxes -- an example

Consider Maxwell’s equations

\[
\begin{bmatrix}
\varepsilon(x) & 0 \\
0 & \mu(x)
\end{bmatrix}
\frac{\partial}{\partial t}
\begin{bmatrix}
E \\
H
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\frac{\partial}{\partial x}
\begin{bmatrix}
E \\
H
\end{bmatrix}
= 0.
\]

The exact same approach leads to

\[
H^* = \frac{1}{\{Z\}} \left( \{ZH\} + \frac{1}{2} [E] \right),
\quad
E^* = \frac{1}{\{Y\}} \left( \{YE\} + \frac{1}{2} [H] \right),
\]

Now assume smooth materials:

\[
H^* = \{H\} + \frac{Y}{2} [E],
\quad
E^* = \{E\} + \frac{Z}{2} [H],
\]

We have recovered the LF flux!

\[
\begin{align*}
Z^\pm &= \sqrt{\frac{\mu^\pm}{\varepsilon^\pm}} = (Y^\pm)^{-1}, \\
Y &= \frac{Z}{\varepsilon} = \frac{1}{\sqrt{\varepsilon \mu}} = c
\end{align*}
\]
Lets move on

At this point we have a good understanding of stability for linear problems -- through the flux.

Lets now look at accuracy in more detail.

Recall

$$\Omega \simeq \Omega_h = \bigcup_{k=1}^{K} D^k,$$

$$u(x, t) \simeq u_h(x, t) = \bigoplus_{k=1}^{K} u_h^k(x, t),$$

we assume the local solution to be

$$x \in D^k = [x_l^k, x_r^k] : u_h^k(x, t) = \sum_{n=1}^{N_p} \hat{u}_n^k(t) \psi_n(x) = \sum_{i=1}^{N_p} u_h^k(x_i^k, t) \ell_i^k(x).$$

modal basis    nodal basis
Local approximation

To simplify matters, introduce local affine mapping

\[ x \in D^k : x(r) = x_i^k + \frac{1 + r}{2} h^k, \quad h^k = x_r^k - x_i^k, \quad r \in [-1, 1] \]

Consider first the modal expansion

\[ u_h(r) = \sum_{n=1}^{N_p} \hat{u}_n \psi_n(r). \]

We immediately have

\[ (u(r), \psi_m(r))_I = \sum_{n=1}^{N_p} \hat{u}_n (\psi_n(r), \psi_m(r))_I, \quad (u, v)_I = \int_{-1}^{1} uv \, dx. \]

or

\[ M\hat{u} = u, \]

\[ M_{ij} = (\psi_i, \psi_j)_I, \quad \hat{u} = [\hat{u}_1, \ldots, \hat{u}_{N_p}]^T, \quad u_i = (u, \psi_i)_I, \]
Local approximation

Consider the most straightforward choice

\[ \psi_n(r) = r^{n-1} \]

... and look at the condition number of \( M \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa(M) )</td>
<td>1.4e+1</td>
<td>3.6e+2</td>
<td>3.1e+5</td>
<td>3.0e+11</td>
</tr>
</tbody>
</table>

This does not look like a robust choice for high \( N \)!

But why? - note that

\[
M_{ij} = \frac{1}{i + j - 1} \left[ 1 + (-1)^{i+j} \right],
\]

Linear dependence for high \( N \) \((i,j)\)
Local approximation

With this understanding, look for a better basis

Idea - make $M$ diagonal - an orthonormal basis

The **Legendre Polynomials**

$$\psi_n(r) = \tilde{P}_{n-1}(r) = \frac{P_{n-1}(r)}{\sqrt{\gamma_{n-1}}}, \quad \gamma_n = \frac{2}{2n + 1}$$

Evaluated as

$$r\tilde{P}_n(r) = a_n\tilde{P}_{n-1}(r) + a_{n+1}\tilde{P}_{n+1}(r),$$

$$a_n = \sqrt{\frac{n^2}{(2n + 1)(2n - 1)}},$$

$$\psi_1(r) = \tilde{P}_0(r) = \frac{1}{\sqrt{2}}, \quad \psi_2(r) = \tilde{P}_1(r) = \sqrt{\frac{3}{2}}r.$$
Local approximation

So we could evaluate these as

\[ \hat{u}_n = (u, \psi_n)_{|} = (u, \tilde{P}_{n-1})_{|}. \quad \text{or} \quad \hat{u}_n \simeq \sum_{i=1}^{N_p} u(r_i) \tilde{P}_{n-1}(r_i) w_i, \]

We instead require that

\[ u(\xi_i) = \sum_{n=1}^{N_p} \hat{u}_n \tilde{P}_{n-1}(\xi_i), \quad \text{for some} \quad \xi_i \]

This yields

\[ V \hat{u} = u, \]

\[ V_{ij} = \tilde{P}_{j-1}(\xi_i), \quad \hat{u}_i = \hat{u}_i, \quad u_i = u(\xi_i). \]

\( V \) is a Vandermonde matrix.
Local approximation

Recall first that we have
\[ u(r) \simeq u_h(r) = \sum_{i=1}^{N_p} u(\xi_i) \ell_i(r). \]
and
\[ u(r) \simeq u_h(r) = \sum_{n=1}^{N_p} \hat{u}_n \tilde{P}_{n-1}(r), \]

This immediately implies
\[ \mathcal{V}^T \ell(r) = \tilde{P}(r), \]
or
\[ \ell_i(r) = \frac{\text{Det}[\mathcal{V}^T(:, 1), \mathcal{V}^T(:, 2), \ldots, \tilde{P}(r), \mathcal{V}^T(:, i+1), \ldots, \mathcal{V}^T(:, N_p)]}{\text{Det}(\mathcal{V}^T)}. \]

so we should choose the point to maximize the determinant -- the solution is

zeros of \[ f(r) = (1 - r^2) \tilde{P}_N'(r). \]
Local approximation

This problem has an exact solution

\[ f(r) = (1 - r^2)\tilde{P}_N'(r). \]

also known as the Gauss-Lobatto quadrature points

Note: These happen to also be quadrature points which we could use to evaluate

\[ \hat{u}_n \simeq \sum_{i=1}^{N_p} u(r_i)\tilde{P}_{n-1}(r_i)w_i, \]

However, they were only chosen to be good for interpolation -- this is essential in 2D/3D
Local approximation - an example

So does it really matter? -- consider $V$

*a)*

$$\begin{bmatrix}
1.00 & -1.00 & 1.00 & -1.00 & 1.00 & -1.00 & 1.00 \\
1.00 & -0.67 & 0.44 & -0.30 & 0.20 & -0.13 & 0.09 \\
1.00 & -0.33 & 0.11 & -0.04 & 0.01 & -0.00 & 0.00 \\
1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
1.00 & 0.33 & 0.11 & 0.04 & 0.01 & 0.00 & 0.00 \\
1.00 & 0.67 & 0.44 & 0.30 & 0.20 & 0.13 & 0.09 \\
1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00
\end{bmatrix}$$

*b)*

$$\begin{bmatrix}
1.00 & -1.00 & 1.00 & -1.00 & 1.00 & -1.00 & 1.00 \\
1.00 & -0.83 & 0.69 & -0.57 & 0.48 & -0.39 & 0.33 \\
1.00 & -0.47 & 0.22 & -0.10 & 0.05 & -0.02 & 0.01 \\
1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
1.00 & 0.47 & 0.22 & 0.10 & 0.05 & 0.02 & 0.01 \\
1.00 & 0.83 & 0.69 & 0.57 & 0.48 & 0.39 & 0.33 \\
1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00
\end{bmatrix}$$

*c)*

$$\begin{bmatrix}
0.71 & -1.22 & 1.58 & -1.87 & 2.12 & -2.35 & 2.55 \\
0.71 & -0.82 & 0.26 & 0.49 & -0.91 & 0.72 & -0.04 \\
0.71 & -0.41 & -0.53 & 0.76 & 0.03 & -0.78 & 0.49 \\
0.71 & 0.00 & -0.79 & 0.00 & 0.80 & 0.00 & -0.80 \\
0.71 & 0.41 & -0.53 & -0.76 & 0.03 & 0.78 & 0.49 \\
0.71 & 0.82 & 0.26 & -0.49 & -0.91 & -0.72 & -0.04 \\
0.71 & 1.22 & 1.58 & 1.87 & 2.12 & 2.35 & 2.55
\end{bmatrix}$$

*d)*

$$\begin{bmatrix}
0.71 & -1.22 & 1.58 & -1.87 & 2.12 & -2.35 & 2.55 \\
0.71 & -1.02 & 0.84 & -0.35 & -0.28 & 0.81 & -1.06 \\
0.71 & -0.57 & -0.27 & 0.83 & -0.50 & -0.37 & 0.85 \\
0.71 & 0.00 & -0.79 & 0.00 & 0.80 & 0.00 & -0.80 \\
0.71 & 0.57 & -0.27 & -0.83 & -0.50 & 0.37 & 0.85 \\
0.71 & 1.02 & 0.84 & 0.35 & -0.29 & -0.81 & -1.06 \\
0.71 & 1.22 & 1.58 & 1.87 & 2.12 & 2.35 & 2.55
\end{bmatrix}$$

**Bad points**

**Good points**

**Bad basis**

**Good basis**
Local approximation - an example

It matters a great deal

It is the basic structure that really matters
Let's summarize this

So we have the local approximations

\[ u(r) \simeq u_h(r) = \sum_{n=1}^{N_p} \hat{u}_n \tilde{P}_{n-1}(r) = \sum_{i=1}^{N_p} u(r_i) \ell_i(r), \]

\[ u = \mathcal{V} \hat{u}, \quad \mathcal{V}^T \ell(r) = \tilde{P}(r), \quad \mathcal{V}_{i,j} = \tilde{P}_j(r_i). \]

and \( r_i \) are the Legendre Gauss Lobatto points:

zeros of \[ f(r) = (1 - r^2) \tilde{P}_N'(r). \]

This leads to a robust way of computing/evaluating a high-order polynomial approximation.

but is it accurate?
A second look at approximation

We will need a little more notation

Regular energy norms

\[ \|u\|_{\Omega}^2 = \int_{\Omega} u^2 \, dx \quad \|u\|_{\Omega,h}^2 = \sum_{k=1}^{K} \|u\|_{D_k}^2, \quad \|u\|_{D_k}^2 = \int_{D_k} u^2 \, dx. \]

Sobolev norms

\[ \|u\|_{\Omega,q}^2 = \sum_{|\alpha|=0}^{q} \|u^{(\alpha)}\|_{\Omega}^2, \quad \|u\|_{\Omega,q,h}^2 = \sum_{k=1}^{K} \|u\|_{D_k}^2, \quad \|u\|_{D_k}^2 = \sum_{|\alpha|=0}^{q} \|u^{(\alpha)}\|_{D_k}^2. \]

Semi-norms

\[ |u|_{\Omega,q,h}^2 = \sum_{k=1}^{K} |u|_{D_k}^2, \quad |u|_{D_k}^2 = \sum_{|\alpha|=q} \|u^{(\alpha)}\|_{D_k}^2. \]
Recall

\[ \Omega \simeq \Omega_h = \bigcup_{k=1}^{K} D^k, \quad u(x, t) \simeq u_h(x, t) = \bigoplus_{k=1}^{K} u^k_h(x, t), \]

we assume the local solution to be

\[ x \in D^k = [x^k_l, x^k_r] : u^k_h(x, t) = \sum_{n=1}^{N_p} \hat{u}^k_n(t) \psi_n(x) = \sum_{i=1}^{N_p} u^k_h(x^k_i, t) \ell^k_i(x). \]

The question is in what sense is \[ u(x, t) \simeq u_h(x, t) \]

We have observed improved accuracy in two ways
- Increase K/decrease h
- Increase N
Approximation theory

Let us assume all elements have size \( h \) and consider

\[ v(r) = u(hr) = u(x); \]

We consider expansions as

\[ v_h(r) = \sum_{n=0}^{N} \hat{v}_n \tilde{P}_n(r), \quad \tilde{P}_n(r) = \frac{P_n(r)}{\sqrt{\gamma_n}}, \quad \gamma_n = \frac{2}{2n+1}. \quad \tilde{v}_n = \int_1 v(r) \tilde{P}_n(r) \, dr. \]
Approximation theory

Let us assume all elements have size \( h \) and consider

\[
v(r) = u(hr) = u(x);
\]

We consider expansions as

\[
v_h(r) = \sum_{n=0}^{N} \hat{v}_n \tilde{P}_n(r), \quad \tilde{P}_n(r) = \frac{P_n(r)}{\sqrt{\gamma_n}}, \quad \gamma_n = \frac{2}{2n+1}. \quad \tilde{v}_n = \int_1^r v(r) \tilde{P}_n(r) \, dr.
\]

**Theorem 4.1.** Assume that \( v \in H^p(1) \) and that \( v_h \) represents a polynomial projection of order \( N \). Then

\[
\|v - v_h\|_{1,q} \leq N^{p-p} |v|_{1,p},
\]

where

\[
\rho = \begin{cases} \frac{3}{2} q, & 0 \leq q \leq 1 \\ 2q - \frac{1}{2}, & q \geq 1 \end{cases}
\]

and \( 0 \leq q \leq p \).
A sharper result can be obtained by using

**Lemma 4.4.** If \( v \in H^p(I), \ p \geq 1 \) then

\[
\| v^{(q)} - v_h^{(q)} \|_{1,0} \leq \left[ \frac{(N + 1 - \sigma)!}{(N + 1 + \sigma - 4q)!} \right]^{1/2} |v|_{1,\sigma},
\]

where \( \sigma = \min(N + 1, p) \) and \( q \leq p \).

Note that in the limit of \( N \gg p \) we recover

\[
\| v^{(q)} - v_h^{(q)} \|_{1,0} \leq N^{2q-p} |v|_{1,p},
\]

A minor issue arises -- these results are based on projections and we are using interpolations.
Approximation theory

We consider

\[ v_h(r) = \sum_{n=0}^{N} \tilde{v}_n \tilde{P}_n(r), \quad \tilde{v}_h(r) = \sum_{n=0}^{N} \tilde{v}_n \tilde{P}_n(r), \quad \upsilon = \mathcal{V} \tilde{\upsilon}, \]

Compare the two

\[ (\mathcal{V} \tilde{\upsilon})_i = v_h(r_i) = \sum_{n=0}^{\infty} \tilde{v}_n \tilde{P}_n(r_i) = \sum_{n=0}^{N} \tilde{v}_n \tilde{P}_n(r_i) + \sum_{n=N+1}^{\infty} \tilde{v}_n \tilde{P}_n(r_i), \]

\[ \mathcal{V} \tilde{\upsilon} = \mathcal{V} \tilde{\upsilon} + \sum_{n=N+1}^{\infty} \tilde{v}_n \tilde{P}_n(r), \]

\[ v_h(r) = \tilde{v}_h(r) + \tilde{P}_T(r) \mathcal{V}^{-1} \sum_{n=N+1}^{\infty} \tilde{v}_n \tilde{P}_n(r). \]
Approximation theory

Consider this term

$$\tilde{P}^T (r) \mathcal{V}^{-1} \sum_{n=N+1}^{\infty} \tilde{v}_n \tilde{P}_n(r) = \sum_{n=N+1}^{\infty} \tilde{v}_n \left( \tilde{P}^T (r) \mathcal{V}^{-1} \tilde{P}_n(r) \right),$$

$$\tilde{P}^T (r) \mathcal{V}^{-1} \tilde{P}_n(r) = \sum_{l=0}^{N} \tilde{p}_l \tilde{P}_l(r), \quad \mathcal{V} \tilde{p} = \tilde{P}_n(r),$$

Caused by interpolation of high-frequency unresolved modes

Aliasing

Caused by the grid
Approximation theory

This has a some impact on the accuracy

**Theorem 4.5.** Assume that $v \in H^p(I)$, $p > \frac{1}{2}$, and that $v_h$ represents a polynomial interpolation of order $N$. Then

$$\|v - v_h\|_{1,q} \leq N^{2q-p+1/2} |v|_{1,p},$$

where $0 \leq q \leq p$. 

Approximation theory

This has a some impact on the accuracy

**Theorem 4.5.** Assume that $v \in H^p(I)$, $p > \frac{1}{2}$, and that $v_h$ represents a polynomial interpolation of order $N$. Then

$$\|v - v_h\|_{l,q} \leq N^{2q-p+1/2}|v|_{l,p},$$

where $0 \leq q \leq p$.

To also account for the cell size we have

**Theorem 4.7.** Assume that $u \in H^p(D^k)$ and that $u_h$ represents a piecewise polynomial approximation of order $N$. Then

$$\|u - u_h\|_{\Omega,q,h} \leq Ch^{\sigma-q}|u|_{\Omega,\sigma,h},$$

for $0 \leq q \leq \sigma$, and $\sigma = \min(N + 1, p)$. 
Combining everything, we have the general result

**Theorem 4.8.** Assume that \( u \in H^p(D^k), \ p > 1/2, \) and that \( u_h \) represents a piecewise polynomial interpolation of order \( N \). Then

\[
\|u - u_h\|_{\Omega,q,h} \leq C \frac{h^{\sigma-q}}{N^{p-2q-1/2}} |u|_{\Omega,\sigma,h},
\]

for \( 0 \leq q \leq \sigma \), and \( \sigma = \min(N + 1, p) \).

with \( h = \max_k h^k \)
Lets summarize

**Fluxes:**
• For linear systems, we can derive exact upwind fluxes using Rankine-Hugoniot conditions.

**Accuracy:**
• Legendre polynomials are the right basis
• We must use good interpolation points
• Local accuracy depends on elementwise smoothness
• Aliasing appears due to the grid but is under control
• For smooth problems, we have a spectral method
• Convergence can be recovered in two ways
  • Increase N
  • Decrease h