Problem Solving Session 6:

Roe’s matrix and TVD methods

Exercise 6.1 In this exercise we compute a Roe matrix $\hat{A}$ for the isothermal flow equations

$$
\begin{bmatrix}
\rho \\
m
\end{bmatrix}_t + \begin{bmatrix}
m \\
\frac{m}{a^2 \rho + m^2 / \rho}
\end{bmatrix}_x = 0
$$

(1)

Using the usual notation of $u_t + f(u)_x = 0$ we have

$$
u = \begin{bmatrix}
\rho \\
m
\end{bmatrix}, \quad f(u) = \begin{bmatrix}
m \\
\frac{m}{a^2 \rho + m^2 / \rho}
\end{bmatrix}
$$

(2)

Here $m$ and $\rho$ are the momentum mass densities respectively. (a) Let $z$ be given by $z = [\alpha \quad \beta]^T = \rho^{-\frac{1}{2}} u$, and define $\varphi(z) = f(u(z))$. Express $u$ as a function of $z$ (i.e., of $\alpha$ and $\beta$ only), write $\varphi$ explicitly as a function of $\alpha$ and $\beta$ only), and compute $f'(u)$, $\varphi'(z)$ and $\frac{d\mu}{dz}(z)$. (b) Suppose that $z_l$ and $z_r$ satisfy $u_l = u(z_l)$ and $u_r = u(z_r)$, and that $z$ has the form

$$
z = z(u) = z_l + (z_r - z_l) \mu, \mu \in (0,1)
$$

(3)

Justify the equality

$$
f(u_r) - f(u_l) = \int_0^1 \varphi'(z(\mu)) d\mu (z_r - z_l) := \hat{C}(z_r - z_l)
$$

(4)

and write $\hat{C}$ explicitly. (c) Integrate the function $\frac{d\mu}{dz}(u(z(\mu)))$ over $\mu \in (0,1)$, where $z(\mu)$ is given by (3), to find a matrix $\hat{B}$ such that $u_r - u_l = \hat{B}(z_r - z_l)$. (d) Verify that $\hat{A} = \hat{C}\hat{B}^{-1}$ satisfies Roe’s three conditions.

Exercise 6.2 For a given Roe matrix $\hat{A}$, the numerical flux $F$ may be given by either of the expressions below

i) $F(u_l, u_r) = f(u_r) - \hat{A}^+(u_r - u_l)$

ii) $F(u_l, u_r) = f(u_l) + \hat{A}^-(u_r - u_l)$

iii) $F(u_l, u_r) = \frac{1}{2} [f(u_l) + f(u_r)] - \frac{1}{2} \left| \hat{A} \right| (u_l - u_r)$

(a) Are these fluxes equivalent? Compare them with the numerical flux of Godunov’s method for linear systems. (b) Compute the matrices $\hat{A}^+$ and $\left| \hat{A} \right|$, where $\hat{A}$ is the Roe matrix you obtained in the previous exercise.

Exercise 6.3 For a grid function $U$, we define the total variation $TV(U)$ of $U$ by

$$
TV(U) = \sum_{j=-\infty}^{\infty} |U_{j+1} - U_j|
$$

(5)

We say that a method $U^{n+1} = H_k(U^n)$ is total variation diminishing (TVD), if for each grid function $U^n$ there exists some $k_0$ such that for $0 < k \leq k_0$, the functions $U^{n+1} = H_k(U^n)$ produced by the scheme satisfy

$$
TV(U^{n+1}) \leq TV(U^n) \quad \forall n
$$

(6)

(a) Use the definition above to show that the Lax-Friedrichs scheme for the scalar conservation law $u_t + f(u)_x = 0$ is TVD. (Hint: first show that if a grid function has finite total variation, then it is bounded.) In this exercise you may assume that $U^0$ has compact support. (b) Use the result in (a) to deduce that the Lax-Friedrichs scheme is stable in the sense of total variation. (TV-stable).