Problem Solving Session 3:

Conservative methods

Exercise 3.1 In the previous exercises we have seen that for discontinuous solutions difficulties arise even on linear problems. For nonlinear problems additional issues arise. In this exercise set we will address these issues. Take the burgers equation in the quasilinear form

\[ u_t + uu_x = 0 \]  

(1)

A natural finite difference method can be obtained with a minor modification of the upwind method applied to the advection equation, assuming \( U^n_j \geq 0 \) for all \( j, n \),

\[ U^{n+1}_j = U^n_j - \frac{k}{h} U^n_j (U^n_j - U^n_{j-1}) \]  

(2)

This method will converge on smooth solutions. (a) Implement the method in Matlab/Octave, solve Eq. (1) to \( T = 1 \) on the interval \([-1, 1]\) with \( k = 0.5h \) for \( h = 0.01 \) using the initial condition

\[ u(x, 0) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases} \]  

(3)

(b) Is the solution obtained a weak solution? Is it an entropy solution? (c) Now use the following initial condition in your code

\[ u(x, 0) = \begin{cases} 1.2 & x < 0 \\ 0.4 & x \geq 0 \end{cases} \]  

(4)

(d) Compare the solution to the true solution which you can construct considering characteristics and shocks as in the first exercise session. Comment.

Exercise 3.2 With the upwind inspired discretization presented in exercise 3.1 we did not have much luck. (a) Implement the generalization of the Lax-Friedrichs method Eq. 5 on the burgers equation 1 with the initial conditions Eq. 3 and Eq. 4.

\[ U^{n+1}_j = \frac{1}{2} (U^n_{j-1} + U^n_{j+1}) - \frac{k}{2h} (f(U^n_{j+1}) - f(U^n_{j-1})) \]  

(5)

(b) Do we now converge to a weak solution? (c) Is it also the entropy solution? (d) Show that the generalization of the Lax-Friedrichs method to nonlinear conservation laws can be written in conservation form. (e) With a method on conservative form, are we guaranteed to arrive at a weak solution? (f) Are we guaranteed to arrive at an entropy solution?

Exercise 3.3 Important conceptual questions for the three exercises: (a) When is a numerical method said to be conservative? (b) What is the benefit of having a numerical method in conservative form? (Hint: Take a look at the Lax-Wendroff Theorem) (c) For a given equation with any scheme on conservative form, are we guaranteed that the weak solution obtained satisfy the entropy condition?