Let us begin the discussion of ODE's

\[
\frac{du}{dt} = f(u, v) \quad u(c) = u_0
\]

\[
u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \quad t \in [c, T] \quad q : \mathbb{R}^m \rightarrow \mathbb{R}^m
\]

This can be the system itself - or we can consider

\[
q_1 u + q_2 u' + \ldots + q_m u^{(m)} = 0
\]

\[
v(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ -q_1 & -q_2 & \cdots & -q_m \\ \vdots & \vdots & \ddots & \vdots \\ -q_{m-1} & -q_m & \cdots & -q_1 \end{bmatrix}
\]

\[
\Rightarrow \quad \frac{dv}{dt} = Av
\]

A - Companion Matrix.

So we can focus on

\[
\frac{dv}{dt} = f(v, t)
\]

**Question** - Does the solution exist?

Is it unique?

**Example**

\[
x = 1 + x^2 \quad ?
\]

\[
x(c) = 0
\]

Solution only exist \( t \in \mathbb{R} \)
To Settle this, let us assume

1) \( f(u, v, t) \) is continuous in \( t \)

\[ \Rightarrow \quad u(t) = u(0) + \int_0^t f(u(s), v) \, ds \quad \text{is a solution} \]

\[
\frac{du}{dt} = f(u(t), v) = f(u(t), v(t)) + \int_0^t \left( \frac{d}{dt} \frac{d}{ds} f(u(s), v(s)) \right) \, ds \\
\Rightarrow \quad \frac{du}{dt} = f(u(t), v) \quad \text{Solution exists}
\]

2) \( f(u, v, t) \) is Lipschitz in \( u \)

\[ \Rightarrow \quad \| f(u, v, t) - f(v, v, t) \| \leq L \| u - v \| \]

- \( \| u \| \) some norm, e.g. \( \| u \|_2 = \sqrt{\sum_i u_i^2} \), \( \| u \|_\infty = \max_i |u_i| \)

\[
\begin{align*}
\| f(u) \| & = \| f(v) \| \\
\| u \| & = \| v \|
\end{align*}
\]

If \( f(u) \) is strongly differentiable in \( u \) then

\[
\max_{u, t} \left| \frac{\partial f}{\partial u} \right| = \lambda \\
\Rightarrow \quad \frac{1}{\| u - v \|} \| f(u) - f(v) \| \leq \left| \frac{\partial f}{\partial u} \right|
\]
Ex

\[ x^2 + 1 - x - 1 = 1x^2 - x^1 + L \]

\[ \text{L depends on } x \text{ - no bound} \]

However, solution exists to \( L \in \frac{3}{2}, \frac{5}{3} \)

---

\[ \text{Take } t_m = \frac{m}{L}, \quad 0 \leq \theta < 1 \]

\[ u(t) = u(t_m) + \int_{t_m}^{t} f(u(s), s) \, ds \quad t \in [t_m, t_m + L] \]

**Solution by iteration**

Now consider two solutions

\[ u_{sol_1}(t) = u(t_m) + \int_{t_m}^{t} f(u_k(s), s) \, ds \]

\[ u_{sol_2}(t) = u(t_m) + \int_{t_m}^{t} f(u_k(s), s) \, ds \]

\[ \| u_{sol_1} - u_{sol_2} \| \leq \int_{t_m}^{t_m + L} \| f(u_k(s), s) \| \, ds \leq \int_{t_m}^{t_m + L} \| f(u_k(s), s) \| \, ds \leq \frac{d}{L} \]

\[ t_{m+1} - t_m = \frac{d}{L} \]

\[ \Rightarrow \quad \| u_{sol_1} - u_{sol_2} \| \leq \frac{d}{2} \]

Since \( \frac{d}{2} \), \( \| u_{sol_1} - u_{sol_2} \| \to 0 \) as \( k \to \infty \)

\[ \Rightarrow \quad \text{Solution is unique} \]
Theorem - Picard's Theorem

If \( f(t, x) \) is Lipschitz in \( x \) and continuous in \( t \)

Then

1. A unique solution exists - \( u(t) \in C \)
2. \( u(t) = u(0) + \int_0^t f(u(s)) \, ds \)

One can furthermore prove that

\[ \| z - u(t) \| \leq e^{\lambda t} \| z - u(0) \| \]

To see this consider

\[ \frac{d}{dt} \| z(t) - u(t) \| \leq L \| z(t) - u(t) \| \]

* \( \text{Exp}(\lambda t) \Rightarrow \frac{d}{dt}(\text{Exp}(\lambda t) \| z(t) - u(t) \|) \leq 0 \)

\[ \Rightarrow \text{Exp}(\lambda t) \| z(t) - u(t) \| \leq \text{Exp}(\lambda t) \| z_0 - u_0 \| \]

\[ \Rightarrow \| z(t) - u(t) \| \leq \text{Exp}(\lambda t) \| z_0 - u_0 \| \]

**Why is this important?**

-Because a computer is noisy!

Going forward we will always assume

Picard's Theorem holds
Let us now return to the simplest scheme

\[ U^{n+1} = U^n + hf^n \]

- Forward Euler Method

This is used to solve \[ u_t = f(u, t) \quad u(0) = u_0. \]

What we care about is **convergence**

\[ \lim_{h \to 0} \max_{n=0}^{N} \| U^n - u(t^n) \| = 0 \]

- \( O \)

**Scheme**

\[ U^{n+1} = U^n + hf^n \]

**Exact Solution**

\[ u(t^n + h) - u(t^n) - hf(u(t^n)) = u(t^n) + hf^n + \frac{h^2}{2} \frac{d}{dt} u(t^n) - h^2 f(u(t^n)) \]

\[ = O(h^2) \]

\[ \Rightarrow u(t^{n+1}) = u(t^n) + hf(u(t^n)) + O(h^2) \]

**Error Equation**

\[ \varepsilon^{n+1} = U^{n+1} - u(t^{n+1}) = \varepsilon^n + hf(u(t^n)) + O(h^2) \]

\[ \Rightarrow \| \varepsilon^{n+1} \| = \| \varepsilon^n \| + h \| f(u(t^n)) - f(u(t^n)) \| + O(h^2) \]

\[ \leq \| \varepsilon^n \| + h \| \varepsilon^n \| = (1 + hL) \| \varepsilon^n \| + O(h^2) \]

**Claim.**

\[ \| \varepsilon^n \| \leq C \frac{h}{L} \left( (1 + nh)^2 - 1 \right) \quad n = 0, 1, \ldots \]

**Proof by induction.**

- \( n = 0 \) - trivial.
\[ \| E_n \| \leq C \left( 1 + L h \right)^{1/2} \left[ (1 + hL)^{-1} - 1 \right] + O(h^2) \]
\[ \leq C \frac{h}{L} (1 + hL)^{-1} - C(1 + hL)^{1/2} + Ch^2 \]
\[ \leq C \frac{h}{L} [(1 + hL)^{-1} - 1] \]

Since \( hL > 0 \Rightarrow 1 + hL \approx e^{hL} \Rightarrow \)

\[ \| E_n \| \leq \frac{C}{L} [e^{1L} - 1] \Rightarrow C \ll 1 \]

**The Forward Euler is Globally Convergent**

And \( \| E_n \| \leq C(T)h \Rightarrow 1^{\text{st}} \text{ Order Accurate} \)


- **Local Truncation Error** \( = O(h^2) \)
  Insert exact solution into scheme

- **Global Error** = Error over time \( \Rightarrow \frac{1}{h} \) steps
  - Accumulates \( \frac{1}{h} - h^2 - h \) - Global error
  Euler is 1^{\text{st}} order scheme

\[ \text{Result above looks like error estimate} \]

\[ \text{Ex: } u = -100u, \quad u(x) = 1 \Rightarrow L = 100, \quad C = u^{\frac{3}{2}}, \quad \text{and } C = 100^2 \]
\[ \Rightarrow \| E_n \| \leq 2.7 \cdot 10^{-45}h \Rightarrow T = 1. \]

**Use Less** - Caused by \( L \) is too crude
\[ Y' = 2t \quad Y(0) = 0 \Rightarrow Y(t) = t^2 \]

\[ Y^2 = Y' + 2hY = 0 \]
\[ Y^2 = Y' + 2hY = 3h^2 \]
\[ Y^2 = Y' + 2hY = 6h^2 \]

\[ \Rightarrow Y^2 = h^2 \frac{2k}{n} = h^2 (k-1) \]

At \( T = 1 \)
\[ h = \frac{1}{n} \Rightarrow \frac{n-1}{n} = 1 - \frac{1}{n} \]

Exact Solution: \( Y = 1 \) \( \sim O(h) \) Diff.

No hope to do better

Spectral Case

\[ u' = -\sqrt{\tan t} - \frac{t}{\cos t} \quad u(0) = 1 \quad u(t) = \cos t - \sin t \]

At \( T = \frac{\pi}{2} \)

Root of \( \exp(t) \cos t = 1 \)

2nd Order

Order Cancellation

Only at this time

\[ Y' = -\frac{tY}{1-t^2} \quad Y(0) = 1 \quad Y = \sqrt{1-t^2} \]

Solution at \( T = 1 \)

\( h^{1/2} \) Lipschitz

\( y(1) = y(1-h) - h \int_0^1 y(1-h) \cdot \frac{1}{t^2} \sqrt{1} \)
\[ \frac{\partial u}{\partial t} = \frac{2u}{(u+2t)^2} \geq 0 \quad \text{if } u > 0 \]

- Initial condition: \( u(0) = 1 \)

- Solution:
  \[ u(t) = \frac{t + \sqrt{1 + 2t^2}}{1 + 2t^2} \]

Test 1: 1-step \( O(h) \)

Test 2: 2-step \( O(h) \)