APMA2811F – Homework 6/Midterm

We shall consider the solution of a nonlinear reaction-diffusion problem which is a simple dynamical model for a nerve impulse.

$$
\begin{aligned}
\frac{\partial y}{\partial t} &= -\frac{1}{\varepsilon}\left(y^3 + ay + b\right) + \sigma\frac{\partial^2 y}{\partial x^2} \qquad (1)\\
\frac{\partial a}{\partial t} &= b + 0.07v + \sigma\frac{\partial^2 a}{\partial x^2}\\
\frac{\partial b}{\partial t} &= (1-a^2)b - a + 0.4y + 0.035v + \sigma\frac{\partial^2 b}{\partial x^2}
\end{aligned}
$$

where

$$x \in [0,1] \ , \ v = \frac{u}{u+0.1} \ , \ u = (y-0.7)(y-1.3) \ ,$$

and (y, a, b) can be considered periodic in space, i.e.,

$$y(0,t) = y(1,t) \ , \ a(0,t) = a(1,t) \ , \ b(0,t) = b(1,t) \ .$$

Initial conditions can be takes to be

$$y(x,0) = 0 \ , \ a(x,0) = -2\cos(2\pi x) \ , \ b(x,0) = 2\sin(2\pi x) \ .$$

The free parameters (ε, σ) control the dynamics by scaling nonlinearity and diffusion.

1) You can use a 2nd order finite difference method to represent the spatial derivatives. Write down the semi-discrete form of the equations.

2) Take $\varepsilon = \sigma = 0.01$, use a standard 3rd order ERK and estimate (by solving computationally until T=0.1) how many grid points are needed in space to accurately/reasonably resolve the spacial dynamics.

Now consider the attached implicit-explicit Runge-Kutta method (IMEX-RK) and its tableau

3) Characterize the explicit RK method, e.g., number of stages, accuracy, stability region etc

4) Characterize the implicit RK method, e.g., number of stages, accuracy, stability (L/A stable), type of IRK etc

Now use the combined IMEX method to solve Eq.(1). For high values of N, the number of grid points, it is the linear parts of the equations that will dominate the stiffness. Thus, it is reasonable to use the implicit RK pair on the linear parts of the equations and the explicit RK pair on the nonlinear parts of the equation.

In the following you can the number of grid points in space to $N = 200$, i.e., $\Delta x = 1/200$.

5) Implement the IMEX-RK method for Eq.(1), including adaptive error control and time-step selection through the embedded error pair. You can use the attached problem description to 'eye-ball' test your code for $N = 32$.

Make sure you pay attention to an efficient implementation of the implicit stage.

6) Study (computationally/by arguments) how the stable timestep of the IMEX-RK scheme scales with ε (take $\varepsilon = 10^{-1}; 10^{-2}; ...; 10^{-6}$) and σ.

7) Take $\varepsilon = \sigma = 10^{-2}$ and study the convergence rate of the combined scheme by computing a highly accurate (in time) reference (or "exact") solution.

8) Take $\varepsilon = 10^{-4}$ and $\sigma = 1/144$. Plot the solution $(y(t), a(t), b(t))$ as a three-dimensional phase portrait for $t \in [0, 10]$.