

AM256 - Homework 9

Due on Thursday, April 24, 2008

Consider Burgers' equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad x \in [-1, 1],$$

where ν is a given constant.

$$u(x, 0) = -\sin(\pi x),$$

and boundary conditions

$$u(-1, t) = 0, \quad u(1, t) = 0.$$

The exact solution to this problem is given as

$$u(x, t) = -\frac{\int_{-\infty}^{\infty} \sin(\pi(x - \eta)) f(x - \eta) \exp\left(-\frac{\eta^2}{4\nu t}\right) d\eta}{\int_{-\infty}^{\infty} f(x - \eta) \exp\left(-\frac{\eta^2}{4\nu t}\right) d\eta},$$

with

$$f(y) = \exp\left(-\frac{\cos \pi y}{2\pi\nu}\right).$$

- 1) Implement the exact solution using a numerical integration formula.
- 2) For $\nu = 1.0$, determine the solution to machine accuracy and plot the solution at $t = 0, 0.25, 0.50, 0.75, 1.0, 1.25, 1.50$. Repeat this for $\nu = 0.1, 0.01, 0.001$.
- 3) Formulate and implement a Fourier Collocation method to solve Burgers' equation as given above. Use a 4th order Runge-Kutta method to integrate in time.

Give an approximate formula for how you expect the time-step to scale with ν and $N - N$ being the length of the Fourier expansion.
- 4) For $\nu = 0.1$, show that the scheme is spectrally convergent.
- 5) Formulate and implement a Chebyshev Collocation method to solve Burgers' equation as given above. Use a 4th order Runge-Kutta method to integrate in time.

Give an approximate formula for how you expect the time-step to scale with ν and N – N being the length of the Chebyshev expansion.

- 6) For $\nu = 0.1$, show the scheme is spectrally convergent.
- 7) Set $\nu = 0.01$ and run the two codes for $N = 32$ – what do you observe? – can you overcome the problems that you see ?
- 8) Set $\nu = 0.001$ and the two codes for $N = 64$ – try to see if you can stabilize the schemes by using a filter.