AM256 - Homework 8

Due on Thursday, April 17, 2008

Consider the following problem

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \quad x \in [-1,1],$$

where a and ν and given constants. The initial condition is

$$u(x,0) = g(x),$$

and boundary conditions

$$u(-1,t) = 0, \quad u(1,t) = h(t).$$

1) Assume that $h(t) = 0$ and derive the conditions on a and ν such that the problem is wellposed in L^2.

2) Assume that we seek solutions using a spectral Galerkin method based on Legendre polynomials. Propose a suitable basis and show that the scheme can be written on the form

$$\mathrm{M}\frac{d\hat{u}}{dt} + a\mathrm{S}\hat{u} = \nu\mathrm{D}\hat{u},$$

Derive the explicit expressions for M, S, and D and show that M is invertible.

3) Assume now that $h(t) = 1$. What changes would you need to make to enable this – please be specific in terms of basis choice etc

4) What can you say about stability of this scheme – is it stable ?

5) Assume now that $h(t)$ is general and derive a Legendre tau-method to solve the problem. Explain the main advantages of the tau method over the Galerkin method.

6) Assume that the solution is now given in terms of a discrete expansion, based on the Legendre Gauss Lobatto points. Propose a Collocation scheme based on these same points.

7) Assume $h(t) = 0$ and prove stability of the collocation scheme.

8) Assume that the approximation is based on Legendre Gauss points but the equation is satisfied on the Legendre Gauss-Lobatto points. Does this change the scheme and its stability – if so, how ?

9) Assume now that the solution is expressed using the Lagrange basis based on the Chebyshev Gauss-Lobatto points and the residual vanishes as the same points. What can you say about stability and convergence in that case ?