Consider the following problem
\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad x \in [-1, 1], \]
where \( a \) and \( \nu \) are given constants. The initial condition is
\[ u(x, 0) = g(x), \]
and boundary conditions
\[ u(-1, t) = 0, \quad u(1, t) = h(t). \]

1.) Assume that \( h(t) = 0 \) and derive the conditions on \( a \) and \( \nu \) such that the problem is well-posed in \( L^2 \).

**Solution:**
We define the operator
\[ \mathcal{L}u = -a u_x + \nu u_{xx}. \]
We assume the solution satisfies \( u(\pm 1, t) = 0 \) and we thus have
\[
\frac{d}{dt} \langle u, u \rangle = \langle u_t, u \rangle + \langle u, u_t \rangle = \langle \mathcal{L}u, u \rangle + \langle u, \mathcal{L}u \rangle = -a \langle u_x, u \rangle + \nu \langle u_{xx}, u \rangle - a \langle u, u_x \rangle + \nu \langle u, u_{xx} \rangle.
\]
\[ \implies -2\nu \|u_x\|^2. \]
Of course, this is only bounded for any \( u \) if \( \nu \geq 0 \). There is no restriction on the real number \( a \).

2.) Assume that we seek solutions using a spectral Galerkin method based on Legendre polynomials. Propose a suitable basis and show that the scheme can be written in the form
\[ M \frac{d\hat{u}}{dt} + aS\hat{u} = \nu D\hat{u}. \]
Derive the explicit expressions for \( M, S, \) and \( D \) and show that \( M \) is invertible.

**Solution:**
We must first derive a basis which satisfies the homogeneous boundary conditions \( u(\pm 1, t) = 0 \). We choose the functions
\[ \phi_n(x) = \begin{cases} 
P_n(x) - P_1(x) & n \text{ odd} \\
P_n(x) - P_0(x) & n \text{ even} \end{cases} \]
which satisfy the conditions $\phi_n(\pm 1) = 0$. (There are no nontrivial constant or linear functions that satisfy the boundary conditions, we assume $n \geq 2$.) We can now form a Galerkin approximation in the standard way:

$$u_N(x,t) = \sum_{j=2}^{N+1} \hat{u}_n(t) \phi_n(x).$$

(2)

We then use (2) in (1) and form the residual:

$$R_N(x,t) = \sum_{j=2}^{N+1} \left( \frac{d\hat{u}_j}{dt} \phi_j + a \hat{u}_j \phi_j' - \nu \hat{u}_j \phi_j'' \right).$$

In the Galerkin formulation, we impose $L^2$-orthogonality of the residual against the expansion functions $\{\phi_n\}_{n=2}^{N+1}$. Let us define

$$M_{ij} = \int_{-1}^{1} \phi_i(x) \phi_j(x) \, dx$$

$$S_{ij} = \int_{-1}^{1} \phi_i(x) \phi_j'(x) \, dx$$

$$D_{ij} = \int_{-1}^{1} \phi_i(x) \phi_j''(x) \, dx.$$ 

Then orthogonalizing the residual $R_N(x,t)$ against $\phi_k(x)$ yields the ODE

$$\sum_{j=2}^{N+1} \frac{d\hat{u}_j}{dt} M_{kj} + a S_{kj} \hat{u}_j - \nu D_{kj} \hat{u}_j = 0.$$ 

By doing this for $k = 2, 3, \ldots, N + 1$, we obtain $N$ ODE’s for the $N$ modal coefficients $\hat{u}_n$. By writing them in matrix-vector format, we obtain

$$M \frac{d\hat{u}}{dt} + a S \hat{u} = \nu D \hat{u},$$

as we sought. We can now work out the entries of each of the matrices. We define the binary indicator

$$b_n = n \mod 2 = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$M_{ij} = \int_{-1}^{1} \phi_i(x) \phi_j(x) \, dx$$

$$= \int_{-1}^{1} (P_i(x) - P_{b_i}(x))(P_j(x) - P_{b_j}(x)) \, dx$$

$$= \langle P_i(x), P_j(x) \rangle - \langle P_i(x), P_{b_j}(x) \rangle - \langle P_{b_i}(x), P_j(x) \rangle + \langle P_{b_i}(x), P_{b_j}(x) \rangle$$

$$= \frac{2}{2j+1} \delta_{ij} + \frac{2}{2b_i+1} \delta_{b_i,b_j}.$$ 

Note that this is a non-sparse matrix: $M_{ij}$ is nonzero if $i - j$ is a multiple of 2. For $S$, we first note that

$$\phi'_n = \begin{cases} P'_n(x) - P_0(x) & n \text{ odd} \\ P'_n(x) & n \text{ even} \end{cases}$$
By direct calculation using the properties of the Legendre polynomials, we can show that

\[ S_{ij} = \int_{-1}^{1} \phi_i(x) \phi'_j(x) \, dx \]

\[ = \int_{-1}^{1} (P_i(x) - P_b(x)) (P'_j(x) - b_j P_b(x)) \, dx \]

\[ = \langle P_i, P'_j \rangle - \langle P_b, P'_j \rangle - b_j \langle P_i, P_b \rangle + b_j \langle P_b, P_b \rangle \]

\[ = 2 \delta_{j+1} \delta_{b+2} - 2 \delta_{b+1} \delta_{j+2} + 2 b_j \delta_{b+1,0}. \]

The second derivative matrix is actually a bit easier since

\[ \phi''_n = P''_n(x). \]

Thus,

\[ D_{ij} = \int_{-1}^{1} \phi_i \phi''_j \, dx \]

\[ = \langle P_i, P''_j \rangle \]

\[ = \frac{2|j(j+1) - i(i+1)|}{2i+1} \delta_{j+1,0}. \]

We have now only to show that the mass matrix is invertible. We know that \( M \) is symmetric since

\[ M_{ij} = \langle \phi_i, \phi_j \rangle = \langle \phi_j, \phi_i \rangle = M_{ji}. \]

Furthermore, we know it’s positive-definite:

\[ u^T M u = \sum_{i,j=2}^{N+1} \bar{u}_i M_{ij} \bar{u}_j \]

\[ = \sum_{i,j=2}^{N+1} \bar{u}_i \bar{u}_j \int_{-1}^{1} \phi_i \phi_j \, dx \]

\[ = \int_{-1}^{1} \sum_{i,j=2}^{N+1} (\bar{u}_i \phi_i) (\bar{u}_j \phi_j) \]

\[ = \int_{-1}^{1} \left( \sum_{n=2}^{N+1} \bar{u}_n \phi_n \right)^2 \, dx \]

\[ > 0, \]

if \( \bar{u} \neq 0 \). A positive-definite matrix cannot be noninvertible (all the eigenvalues are positive). Thus, \( M \) is invertible.

3. Assume now that \( h(t) = 1 \). What changes would you need to make to enable this – please be specific in terms of basis choice, etc.

Solution:
The problem here is that we now have a boundary condition that reads
\[ u(1, t) = 1. \]

Requiring that all the basis functions satisfy this condition will not help since
\[ \phi_n(1) = 1 \neq uN(1, t) = \sum_n \tilde{u}_n \phi_n(1) = \sum_n \tilde{u}_n = 1. \]

Thus, we opt for a different solution: we’ll make the change of variables:
\[ v(x, t) = u(x, t) - \frac{1 + x}{2}. \] (3)

Then
\[ v(-1, t) = u(-1, t) - \frac{1 - 1}{2} = 0, \quad v(1, t) = u(1, t) - \frac{1 + 1}{2} = 0. \]

I.e., the function \( v(x, t) \) satisfies homogeneous Dirichlet boundary conditions at \( x = \pm 1 \), and satisfies the PDE
\[ \frac{\partial v}{\partial t} + a \frac{\partial v}{\partial x} + \frac{\nu}{2} \frac{\partial^2 v}{\partial x^2} = x \in [-1, 1], \] (4)

with initial data
\[ v(x, 0) = g(x, 0) - \frac{1 + x}{2}. \]

With this new PDE for \( v(x, t) \), we can solve it via exactly the same Galerkin method as in problem 2 (using the same basis functions), and then once \( v(x, T) \) has been computed, we can employ the transformation (3) in reverse to obtain \( u \).

4.) What can you say about stability of this scheme - is it stable?

**Solution:** For the homogeneous boundary value problem, there is no concern: via the same integration by parts we did in problem 1, we immediately get stability. However, things are more delicate if \( h(t) = 1 \), but relatively transparent considering the transformation in problem 3. We first consider Galerkin stability of the PDE (4) for \( v(x, t) \):

\[
\frac{d}{dt} \langle v, v \rangle = \langle -a v_x + \nu v_{xx}, v - \frac{a}{2} \rangle + \langle v, -a v_x + \nu v_{xx} - \frac{a}{2} \rangle
\]
\[
= -a \langle v_x, v \rangle - a \langle v_x, v_x \rangle + \nu \langle v_{xx}, v \rangle + \nu \langle v, v_{xx} \rangle - \langle a, v \rangle
\]
\[
\leq -a \langle v_x, v \rangle + a \langle v_x, v_x \rangle + \nu \langle v, v_{xx} \rangle + \nu \langle a, v \rangle
\]
\[
\leq -2 \nu \| u_x \|^2 + a \| v \|^2.
\]

Thus \( \| v \|^2 \leq \alpha t + e^{\alpha t} \), which is stability for \( v(x, t) \). We now only need to notice that
\[
\| u \| = \left\| v + \frac{1 + x}{2} \right\| \leq \| v \| + \left\| \frac{1 + x}{2} \right\| \leq C + \alpha t + e^{\alpha t},
\]

which is \( L^2 \)-stability for \( u(x, t) \). Thus, the Galerkin method will be stable since the PDE is.

5.) Assume now that \( h(t) \) is general and derive a Legendre-Tau method to solve the problem. Explain the main advantages of the Tau method over the Galerkin method.

**Solution:**
We define the expansion functions to be
\[ \phi_n(x) = P_n(x). \]
With this, we expand \( u \) in the first \( N \) Legendre polynomials
\[ u_N(x) = \sum_{n=0}^{N-1} \hat{u}_n \phi_n(x) \]
and then demand \( L^2 \)-orthogonalization of the residual of the PDE against the first \( N - 2 \) expansion functions:
\[
\left( \frac{2}{2n+1} \frac{d}{dt} + a \sum_{p=n+1 \atop p+n \text{ odd}}^{N-1} 2 \hat{u}_p \right) \sum_{p=n+2 \atop p+n \text{ even}}^{N-1} 2 \frac{(p+1) - n(n+1)}{2n+1} \hat{u}_p = 0, \tag{5}
\]
for \( n = 0, 1, \ldots, N - 3 \). We then enforce the boundary conditions with the last two degrees of freedom:
\[
\begin{align*}
  u_N(-1, t) = 0 & \implies \sum_{n=0}^{N-1} \hat{u}_n (-1)^n = 0 \\
  u_N(1, t) = h(t) & \implies \sum_{n=0}^{N-1} \hat{u}_n = h(t)
\end{align*}
\tag{6}
\]
Finally, the initial condition is given by
\[
\hat{u}_n(0) = \frac{1}{\|\phi_n\|^2} \langle g(x), \phi_n(x) \rangle,
\]
for \( n = 2, 3, \ldots, N - 1 \), and equation (6) for \( n = 1, N \).

The differential-algebraic system (5) and (6) is the Legendre-Tau scheme for our PDE with the given boundary conditions.

The major advantage of using the Tau method here instead of the Galerkin method is the ease of satisfying the boundary conditions. We see that equations (6) are much simpler to write down and to implement than what we saw for the Galerkin method above. We don’t have to tailor our Galerkin matrices to the boundary conditions, which itself was a mildly burdensome task. Also, the basis functions don’t change in time, so we don’t have to worry about changing the Galerkin operators in time.

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6.) Assume that the solution is now given in terms of a discrete expansion, based on the Legendre-Gauss-Lobatto points. Propose a collocation scheme based on these same points.

Solution:

We assume that
\[ u_N(x) = \sum_{j=0}^{N} u(x_j, t) l_j(x) = \sum_{j=0}^{N} u_j(t) l_j(x) = \sum_{j=0}^{N} \hat{u}_j P_j(x), \]
where \( x_j \) are the Legendre-Gauss-Lobatto quadrature nodes, and the \( l_j(x) \) are the associated Lagrange polynomials for these points (whose form can be derived from the discrete Legendre expansion and the quadrature rule). We simply use this expansion in the PDE and require the residual to vanish at the Legendre-Gauss-Lobatto points:
\[
\frac{d u_n}{dt} + a \sum_{j=1}^{N} \hat{D}_{nj} u_j = \nu \sum_{j=1}^{N} \hat{D}_{nj}^{(2)} u_j, \quad n = 1, 2, \ldots, N - 1.
\]
We have that the entries of the differentiation matrices are given by

\[ \tilde{D}_{ij} = \frac{dF_j}{dx} \bigg|_{x=x_i}, \]

\[ \tilde{D}_{ij}^{(2)} = \frac{d^2F_j}{dx^2} \bigg|_{x=x_i} = (\tilde{D}^2)_{ij}. \]

This is \( N - 1 \) equations. The last two equations merely impose the boundary conditions:

\[ u_1 = 0 \]

The initial condition is specified by

\[ u_j(t=0) = g(x_j). \]

7.) Assume \( h(t) = 0 \) and prove stability of the collocation scheme.

Solution:

At gridpoints \( \{x_j\}_{j=1}^{N-1} \), the scheme is written

\[ \frac{\partial u_N(x_j)}{\partial t} = -a \frac{\partial u_N(x_j)}{\partial x} + \nu \frac{\partial^2 u_N(x_j)}{\partial x^2}, \]

and we enforce \( u_N(\pm 1, t) = 0 \).

Multiplying the residual statement through by \( w_j u_N(x_j) \) and summing over all gridpoints, we have

\[ \sum_{j=0}^{N-1} \frac{\partial u_N(x_j)}{\partial t} u_N(x_j) w_j = \sum_{j=0}^{N-1} -a \frac{\partial u_N(x_j)}{\partial x} u_N(x_j) w_j + \sum_{j=0}^{N-1} \nu \frac{\partial^2 u_N(x_j)}{\partial x^2} u_N(x_j) \]

Using the exactness of the quadrature for polynomials \( p \in B^{2N-1} \), this reads

\[ \frac{d}{dt} \sum_j \frac{1}{2} u_N(x_j) w_j = -a \int_{-1}^{1} u_N \frac{\partial u_N}{\partial x} dx + \nu \int_{-1}^{1} u_N \frac{\partial^2 u_N}{\partial x^2} dx \]

\[ = -a \int_{-1}^{1} \left( \frac{\partial u_N}{\partial x} \right)^2 dx + \nu \int_{-1}^{1} \frac{\partial u_N}{\partial x} \left. \left( \frac{\partial u_N}{\partial x} \right) \right|_{-1}^{1} \]

\[ = -\nu \int_{-1}^{1} \left( \frac{\partial u_N}{\partial x} \right)^2 dx \]

where we have utilized that \( u_N(\pm 1) = 0 \) to remove all boundary terms. Hence,

\[ \frac{d}{dt} \sum_j u_N^2(x_j) w_j \leq 0 \]

and the scheme is stable in \( L^2_w \).

8.) Assume that the approximation is based on the Legendre-Gauss points, but the equation is satisfied on the Legendre-Gauss-Lobatto points. Does this change the scheme and its stability - if so, how?

Solution:
Let \( \{x_j\}_{j=0}^N \) be the Legendre-Gauss points, \( \{y_j\}_{j=0}^N \) the Legendre-Gauss points, and \( l_{y_j} \) the interpolating Lagrange polynomial corresponding to \( y_j \). Then the approximation \( u_N \) can be written as

\[
    u_N(x) = \sum_{j=0}^N u(y_j, t) l_{y_j}(x)
\]

The first \( N-1 \) equations of the scheme are formed by requiring the residual to vanish at the \( x_j \) grid points, i.e.

\[
    R_N(x_j, t) = \frac{\partial u_N(x_j)}{\partial t} + a \frac{\partial u_N(x_j)}{\partial x} - \nu \frac{\partial^2 u_N(x_j)}{\partial x^2} = 0, \quad j = 1, \ldots, N-1
\]

and the system is closed by enforcing the boundary conditions

\[
    u_N(-1, t) = 0, \quad u_N(1, t) = 0
\]

The main difference in the scheme comes from the non-interpolating behavior of the Lagrange basis

\[
    l_{y_j}(x_j) \neq \delta_{ij}
\]

due to the differing grid points, while the differentiation matrices are

\[
    \tilde{D}_{ij} = \left. \frac{d l_{y_j}}{dx} \right|_{x=x_i}
\]

\[
    \tilde{D}_{ij}^{(2)} = \left. \frac{d^2 l_{y_j}}{dx^2} \right|_{x=x_i}
\]

By multiplying the residual statement by \( u_N(x_j, t)w_j \) and summing, the stability of the scheme can be proven via the same procedure as in problem 7.

9) Assume now that the solution is expressed using the Lagrange basis based on the Chebyshev-Gauss-Lobatto points and the residual vanishes at the same points. What can you say about stability and convergence in this case?

Solution:

Due to the advection term in the equation, the PDE is not well-posed in the Chebyshev norm (see example 8.5 in the text). Due to this, we cannot directly prove stability in the \( L^2_w \) norm.