AM256 - Homework 7

Due on Thursday, April 10, 2008

Consider the following sequence of functions

$$u^{(0)} = \begin{cases} -\cos(\pi x) & -1 \leq x \leq 0 \\ \cos(\pi x) & 0 < x \leq 1 \end{cases} \qquad u^{(i)} = \int_{-1}^{x} u^{(i-1)}(s)\, ds + S$$

where the constant, S, must be chosen such that $u^{(i)} \in C^{i-1}$ for $i \geq 1$. Note also that $u^{(0)} \in L^2$.

1) Derive the 4 functions and plot them.

We shall consider Chebyshev expansions of this sequence to understand the basic approximation properties. The general expansion is given as

$$u(x) \simeq u_N(x) = \sum_{n=0}^{N} \tilde{u}_n T_n(x) \quad ,$$

where $T_n(x)$ is the nth order Chebyshev polynomial.

As we can not recover the exact expansion coefficients for the sequence of functions we shall first use the Gauss-Lobatto quadrature as discussed in the text (Appendix B).

2) As 'exact' expansion coefficients we shall use the discrete ones computed at a very fine grid. Take $N = 1000$ and compute the expansion coefficients using the quadrature and use these as the 'exact' coefficients, i.e., $\hat{u}_n = \tilde{u}_n$ for $n \ll 1000$.

   Plot, as a function of $N < 256$, the absolute value of the expansion coefficients for the 4 test functions. Please comment on the decay of the expansion coefficients and its relation to the smoothness of the function.

3) Use the 'exact' coefficients to estimate the projection error

$$\|u - u_N\|^2_{L^2_w} = \sum_{n=N+1}^{256} \gamma_n \tilde{u}_n^2 \quad ,$$

   for the 4 functions for $N < 128$. What can you say about the connection between regularity and convergence rate?

4) Use the backward recurrence and the 'exact' coefficients to compute the approximation to $\hat{u}'_n$ - i.e. the expansion coefficients for the first

derivative and plot

$$\|u' - u'_N\|^2_{L^2_w} = \sum_{n=N+1}^{256} \gamma_n (\tilde{u}'_n)^2 \ ,$$

for the 4 functions for $n < 128$. What can you say about the connection between regularity and convergence rate?

Let us now consider the interpolation rather than the projection (or rather the approximation of it as in the above). In this case the discrete expansion coefficients are given directly through the quadrature for N (and not some very large value to mimic the projection).

5) Plot, for $N = 8, 16, 32$, the pointwise error of the discrete expansion for the 4 functions. Do you observe anything special such as places where the error vanishes – what points are these ?
6) Plot, as a function of N, the difference $|\hat{u}_n - \tilde{u}_n$ for the 4 functions and for $N = 2^p, p = 2 - 7$. To what do you attribute the error.
7) Use the backward recurrence and the discrete coefficients to compute the approximation to $\tilde{u}'_n$ - i.e. the expansion coefficients for the first derivative and plot

$$\|u' - \mathcal{I}_N u'_N\|_{L^2_w},$$

for the 4 functions for $N = 2^p, p = 2 - 7$. What can you say about the connection between regularity and convergence rate?

We finally consider the Lagrangian form

$$u_N(x) = \sum_{j=0}^{N} u(x_j) g_j(x) \ ,$$

such that a differentiation matrix can be defined by standard ways.

8) Use the differentiation matrix to compute

$$\|u' - \mathcal{I}_N u'_N\|_{L^2_w},$$

for the 4 functions for $N = 2^p, p = 2 - 7$. Compare with the results you obtained in 7) – are they related ?