Consider the following sequence of functions

\[ u^{(0)} = \begin{cases} 
-\cos(\pi x) & -1 \leq x \leq 0 \\
\cos(\pi x) & 0 < x \leq 1 
\end{cases} \quad u^{(i)} = \int_{-1}^{x} u^{(i-1)}(s) \, ds + S, \]

where the constant \( S \) must be chosen so that \( u^{(i)} \in C^{i-1} \) for \( i \geq 1 \). Note also that \( u^{(0)} \in L^2 \).

1.) Derive the 4 functions and plot them

Solution:

With \( u^{(0)} \) as given above, we can directly integrate to get

\[ u^{(1)}(x) = \begin{cases} 
-1 & -1 \leq x \leq 0 \\
\frac{1}{\pi} \sin(\pi x) & 0 < x \leq 1 
\end{cases} \]

\[ u^{(2)}(x) = \begin{cases} 
1 + \cos(\pi x) & -1 \leq x \leq 0 \\
3 - \cos(\pi x) & 0 < x \leq 1 
\end{cases} \]

\[ u^{(3)}(x) = \begin{cases} 
\frac{\pi + \pi x + \sin(\pi x)}{\pi^2} & -1 \leq x \leq 0 \\
\frac{\pi + 3\pi x - \sin(\pi x)}{\pi^2} & 0 < x \leq 1
\end{cases} \]
We shall consider Chebyshev expansions of this sequence to understand the basic approximation properties. The general expansion is given as

\[ u(x) \simeq u_N(x) = \sum_{n=0}^{N} \tilde{u}_n T_n(x), \]

where \( T_n(x) \) is the \( n \)th order Chebyshev polynomial.

As we cannot recover the exact expansion coefficients for the sequence of functions, we shall first use the Guass-Lobatto quadrature as discussed in the text (Appendix B).

2.) As ‘exact’ expansion coefficients we shall use the discrete ones computed at a very fine grid. Take \( N = 1000 \) and compute the expansion coefficients using the quadrature and use these as the ‘exact’ coefficients, i.e. \( \tilde{u}_n = \bar{u}_n \) for \( n \ll 1000 \).

Plot, as a function of \( N < 256 \), the absolute value of the expansion coefficients for the 4 test functions. Please comment on the decay of the expansion coefficients and its relation to the smoothness of the function.

Solution:

Here, the non-zero coefficients are displayed. By measuring the slope we see that \( \tilde{u}_n \) decays as \( \frac{1}{n^q} \), where \( q = 1, 2, 3, 4 \) respectively for each of the four functions. Thus we see that as the smoothness of the function increases, the rate of decay of its expansion coefficients increases.
3.) Use the ‘exact’ coefficients to estimate the projection error

\[ \| u - u_N \|^2_{L^2_w} = \sum_{n=N+1}^{256} \gamma_n \tilde{u}_n^2, \]

for the 4 functions for \( N < 128 \). What can you say about the connection between regularity and convergence rate?

Solution:

\[ \begin{align*}
\text{u}_0, \text{slope} &= 0.6 \\
\text{u}_1, \text{slope} &= -1.5 \\
\text{u}_2, \text{slope} &= -2.6 \\
\text{u}_3, \text{slope} &= -3.4
\end{align*} \]

From theorem 6.1 in the text we predict the projection error \( \| u - u_N \|^2_{L^2_w} \) to decay as \( N^{-p} \), where \( u \in H^{p/2}_{\text{w}}[-1,1] \). For the four functions we have \( p = 0.5, 1.5, 2.5, 3.5 \) (recall the definition of a fractional Sobolev space to verify this). These values are verified numerically by considering the slope of the lines in the figure above. Note that we will need to take the square root of \( \| u - u_N \|^2_{L^2_w} \) to get the correct projection error.

4.) Use the backward recurrence and the ‘exact’ coefficients to compute the approximation to \( \tilde{u}_n \), i.e. the expansion coefficients for the first derivative and plot

\[ \| u' - u'_N \|^2_{L^2_w} = \sum_{n=N+1}^{256} \gamma_n (\tilde{u}_n')^2, \]

for the 4 functions for \( n < 128 \). What can you say about the connection between regularity and
Convergence rate?

Solution:

The recurrence relation, given in equation 5.13 in the text, is

\[ c_n - 1 \hat{u}_{n-1}^{(q)} - \hat{u}_{n+1}^{(q)} = 2n \hat{u}_n^{(q-1)}, \]

where \( c_0 = 2 \) and \( c_n = 1 \) for all other \( n \). Assuming \( \hat{u}_{N+1}^{(q)} = \hat{u}_N^{(q)} = 0 \), we can work backwards to obtain \( \hat{u}_i^{(q)}, i \in \{0,...,N-1\} \). Note that we will need to take the square root of \( \|u' - u_N'\|_{L^2_w}^2 \) to get the correct error estimate.

The convergence rate, as in problem 3, matches the theoretical prediction by decaying as \( N^{-p} \), where \( u' \in H^p_w[-1,1] \). Note that the derivative of function \( u^{(0)} \) is not in \( L^2 \), and therefore its expansion coefficients do not decay.

Let us now consider the interpolation rather than the projection (or rather the approximation of it as in the above.) In this case the discrete expansion coefficients are given directly through the quadrature for \( N \) (and not some very large value to mimic the projection).

5.) Plot for \( N = 8, 16, 32 \) the pointwise error of the discrete expansion for the 4 functions. Do you observe anything special such as places where the error vanishes – what points are there?

Solution: The discrete expansion is calculated by determining the modal coefficients using \( (N +
1)-point Chebyshev-Gauss-Lobatto quadrature for the various values of \( N \). Having done this, the plots below show the absolute error

\[
e_{N}^{(j)}(x) = \left| u^{(j)} - \mathcal{I}_{N}u^{(j)} \right|.
\]

The plots show the error at the 1001-point Chebyshev-Gauss-Lobatto quadrature points.

**Figure 1.** Plot of the pointwise interpolation error \( e_{N}^{(j)}(x) \) for \( N = 8 \). The interpolation points are shown as red dots.

**Figure 2.** Plot of the pointwise interpolation error \( e_{N}^{(j)}(x) \) for \( N = 16 \). The interpolation points are shown as red dots.
Figure 3. Plot of the pointwise interpolation error $e_N^{(j)}(x)$ for $N = 32$. The interpolation points are shown as red dots.

Important things to notice are

- The error is largest in the center of the interval where the interpolation points are more sparse.
- The maximum error decreases as $N$ increases for all functions except $u^{(0)}$, which is not continuous.
- Qualitatively speaking whenever we double $N$, the $L^\infty$ error seems to decrease by $\sim \frac{1}{2}$ for $u^{(1)}$ decrease by $\sim \frac{1}{4}$ for $u^{(2)}$ decrease by $\sim \frac{1}{8}$ for $u^{(3)}$

A qualitative assessment would lead us to believe, for $u \in H^p$, some kind of error estimate of the form

$$\|u - I_N u\|_{L^\infty} \leq C N^{-p}$$

- The error is exactly zero at the interpolation points. We certainly expected this since we're computing an interpolant.

6.) Plot, as a function of $N$, the difference $|\tilde{u}_n - \tilde{u}_n|$ for the 4 functions and for $N = 2^p$, $p = 2, \ldots, 7$. To what do you attribute the error?

Solution: We show this plot in figure 4. Any difference we see must come from the aliasing error, which is the fundamental difference between interpolation and projection.
We note in addition that the rate of decay (indicated by the slopes of the log-log plots) increases with the regularity of the function.

Figure 4. Plot of the maximum modal coefficient deviation between the ‘exact’ \( u_n \) and the discrete \( \tilde{u}_n \).

7.) Use the backward recurrence and the discrete coefficients to compute the approximation to \( u'_n \) - i.e. the expansion coefficients for the first derivative and plot

\[
\| u' - I_N u'_N \|_{L^2_w},
\]

for the 4 functions for \( N = 2^p, \ p = 2, \ldots, 7 \). What can you say about the connection between regularity and convergence rate?

**Solution:** As usual, we do not compute the exact \( L^2 \) error, but rather

\[
\| I_M u' - (I_N u)' \|_{L^2_w},
\]

for \( M = 1000 \). Note that for \( u^{(0)} \), we define \( \left. \frac{du^{(0)}}{dx} \right|_{x=0} = 0 \), even though this is not correct strictly speaking, since \( \frac{du^{(0)}}{dx} \notin L^2 \) and does not have finite derivative at \( x = 0 \). We show the plots in figure 5. Recall from Fourier theory that for expansion in complex exponentials, if we differentiate then we lose an order of convergence. We see a similar phenomenon here as well. By differentiating, we see that our orders of convergence (the slopes of the lines) have decreased. For \( u^{(0)} \), we have no convergence at all, which is not so surprising since \( u^{(0)} \notin H^1 \), so we can’t really take its derivative. \( u^{(j)} \) for \( j > 0 \) exhibits convergence, at faster rates as \( j \) increases. I.e., the more regular the function, the faster the convergence.
We finally consider the Lagrangian form

\[ u_N(x) = \sum_{j=0}^{N} u(x_j) g_j(x), \]

such that the differentiation matrix can be defined by standard ways.

8.) Use the differentiation matrix to compute

\[ \|u' - \mathcal{I}_N u_N'\|_{L^2}, \]

for the 4 functions for \( N = 2^p, p = 2, \ldots, 7 \). Compare with the results you obtained in 7.) – are they related?

**Solution:** We first derive an expression for the differentiation matrix. We know:

\[ \tilde{u}_n = \frac{1}{\gamma_n} \sum_{j=0}^{N} w_j u(x_j) T_n(x_j), \]

where \((x_j, w_j)_{j=0}^{N}\) are the Chebyshev-Gauss-Lobatto nodes and weights. We also know

\[ u_N(x) = \sum_{n=0}^{N} \tilde{u}_n T_n(x). \]

Putting this together, we have

\[
\begin{align*}
    u_N(x) &= \sum_{n=0}^{N} \frac{1}{\gamma_n} \sum_{j=0}^{N} w_j u(x_j) T_n(x_j) T_n(x) \\
    &= \sum_{j=0}^{N} \left( w_j \sum_{n=0}^{N} \frac{1}{\gamma_n} T_n(x_j) T_n(x) \right) u(x_j) \\
    &= \sum_{j=0}^{N} g_j(x) u(x_j),
\end{align*}
\]
which gives us that the expression for the Lagrange polynomials is
\[ g_j(x) = \left( w_j \sum_{n=0}^{N} \frac{1}{\gamma_n} T_n(x_j) T_n(x) \right) \].

Then we can compute
\[ u'_N(x_k) = \sum_{j=0}^{N} g'_j(x_k) u(x_j) = \sum_{j=0}^{N} \left( w_j \sum_{n=0}^{N} \frac{1}{\gamma_n} T_n(x_j) T'_n(x_k) \right) u(x_j) = \sum_{j=0}^{N} D_{kj} u(x_j). \]

Thus, we can define the entries of the differentiation matrix as
\[ D_{kj} = w_j \sum_{n=0}^{N} \frac{1}{\gamma_n} T_n(x_j) T'_n(x_k). \]

Thus, we can explicitly code the differentiation matrix by making use of the recurrence relations to compute \( T_n(x) \) and \( T'_n(x) \). Note that we expect this to be absolutely the same thing as what we did in part 7 since we are doing exactly the same operations, just in a different way. We see confirmation of this in figure 6: the figures and the slopes are identical.

![Plot of the L^2 error between u' and D u_N.](image)

Figure 6. Plot of the \( L^2 \) error between \( u' \) and \( D u_N \).

Just to make sure that everything is as it seems, we plot the quantity
\[ \|(I_N u)' - D u_N'\|_{L^2_w}, \]
in figure 7 below. We see errors of order (machine precision) \( \times \) (the number of operations), which validates the equivalence of the methods in problems 7 and 8.
Figure 7. $L^2$ norm of the difference between the two approaches in problems 7 and 8.