

AM256 - Homework 5

Due on Thursday, Mar 13, 2008

Let us first consider the linear dispersive problem

$$\frac{\partial u}{\partial t} = a(x) \frac{\partial^3 u}{\partial x^3}, \quad x \in [0, 2\pi],$$

and assume that u is a periodic solution with initial conditions $u(x, 0) = g(x)$. $a(x)$ is assumed to be smooth and real.

- 1) Assume first that $a(x)$ is a constant. Derive a Fourier-Galerkin scheme for this equation and prove that it is stable in a semi-discrete sense.
- 2) Assume first that $a(x)$ is a constant. Derive a Fourier-Collocation scheme for this equation and prove that it is stable in a semi-discrete sense.
- 3) Let us now assume that $a(x)$ is variable. Derive (generally) a Fourier-Galerkin scheme and prove that it is stable in a semi-discrete sense. What conditions must be placed on $a(x)$ for this to hold.
- 4) Let us now assume that $a(x)$ is variable but uniformly bounded away from zero, i.e., $0 < |a(x)| < \infty$. Derive a Fourier-Collocation scheme and prove that it is stable in a semi-discrete sense.

Let us now consider the slightly more complicated problem

$$\frac{\partial u}{\partial t} + iu|u|^2 + i \frac{\partial^2 u}{\partial x^2} = 0 \quad ,$$

and assume that u is a periodic solution with initial conditions $u(x, 0) = g(x)$. Here i is the imaginary unit. This equation is known as the non-linear Schrödinger equation and it plays a very important role in nonlinear optics optical fiber communication.

- 5) Assume that a Fourier-Galerkin scheme is derived for this equation. Is it stable ?
- 6) Derive a Fourier-Collocation scheme for this equation. Do you expect it to be stable ? – if, not, please explain what could be the source of an instability and how you could imagine stabilizing the scheme.