Let us first consider the linear dispersive problem
\[ \frac{\partial u}{\partial t} = a(x) \frac{\partial^3 u}{\partial x^3}, \quad x \in [0, 2\pi], \]
and assume that \( u \) is a periodic solution with initial conditions \( u(x, 0) = g(x) \). \( a(x) \) is assumed to be smooth and real.

1) Assume first that \( a(x) \) is a constant. Derive a Fourier-Galerkin scheme for this equation and prove that it is stable in a semi-discrete sense.

2) Assume first that \( a(x) \) is a constant. Derive a Fourier-Collocation scheme for this equation and prove that it is stable in a semi-discrete sense.

3) Let us now assume that \( a(x) \) is variable. Derive (generally) a Fourier-Galerkin scheme and prove that it is stable in a semi-discrete sense. What conditions must be placed on \( a(x) \) for this to hold.

4) Let us now assume that \( a(x) \) is variable but uniformly bounded away from zero, i.e., \( 0 < |a(x)| < \infty \). Derive a Fourier-Collocation scheme and prove that it is stable in a semi-discrete sense.

Let us now consider the slightly more complicated problem
\[ \frac{\partial u}{\partial t} + iu|u|^2 + i \frac{\partial^2 u}{\partial x^2} = 0, \]
and assume that \( u \) is a periodic solution with initial conditions \( u(x, 0) = g(x) \). Here \( i \) is the imaginary unit. This equation is known as the non-linear Schrödinger equation and it plays a very important role in nonlinear optics optical fiber communication.

5) Assume that a Fourier-Galerkin scheme is derived for this equation. Is it stable?

6) Derive a Fourier-Collocation scheme for this equation. Do you expect it to be stable? – if, not, please explain what could be the source of an instability and how you could imagine stabilizing the scheme.