1. Consider the discrete expansion

\[ u_N(x) = \sum_{|n| \leq N/2} \tilde{u}_n \exp(inx), \quad \tilde{u}_n = \frac{1}{c_n N} \sum_{j=0}^{N} u(x_j) \exp(-inx_j), \]

\[ x_j = \frac{2\pi}{N} j, \quad j = 0, \ldots, N - 1, \]

with \( c_n = 1 \) for \( |n| < N/2 \) and \( c_n = 2 \) for \( |n| = N/2 \).

(a) Prove that one can express this as

\[ u_N(x) = \sum_{j=0}^{N-1} u(x_j) g_j(x), \]

derive an explicit expression for \( g_j(x) \) and show that it is a Lagrange polynomial.

(b) The derivative of \( u_N(x) \) at the grid points can be expressed as

\[ \frac{d}{dx} u_N(x_i) = \sum_{j=0}^{N-1} D_{ij} u(x_j). \]

Derive the explicit entries of the differentiation matrix, \( D_{ij} \), and show that this matrix is a skew-symmetric, circulant, Toeplitz matrix.

(c) Show that \( D \) must have at least one zero eigenvalue (HINT: Consider a constant function).

2. Consider the following approximation to the integral

\[ \frac{1}{2\pi} \int_{0}^{2\pi} u(x) \, dx \simeq \frac{1}{N+1} \sum_{j=0}^{N} u(x_j), \quad x_j = \frac{2\pi}{N+1} j, \quad j = 0, \ldots, N. \]

(a) If we define the space of trigonometric functions as

\[ \mathcal{B}_N = \text{span} \{ \exp(inx) \}_{|n| \leq N/2}, \]
prove that the approximation is exact for all \( u(x) \in B_{2N} \).

(b) Confirm the analytic result by computations by choosing \( u(x) \) to
illustrate the accuracy and failure of the approximation

3. Consider the following function

\[
u(x) = \frac{3}{5 - 4 \cos(x)}, \quad x \in [0, 2\pi].\]

We shall explore ways of computing the derivatives of \( u(x) \) using the
continuous and discrete expansion coefficients as well as the differentia-
tion matrix.

(a) Compute the analytic derivatives of \( u(x) \) of up to 4th order, i.e.,
\( u^{(m)}, m = 1, 2, 3, 4. \)

(b) Using the exact continuous expansion coefficients

\[
\hat{u}_n = 2^{-|n|},
\]

compute the max error of \( u^{(m)}_N \) for \( m = 1, 2, 3, 4 \) and for \( N = 2^p, p = 2 - 10. \) Tabulate the results and discuss what you see.

(c) Now repeat this exercise but use the discrete expansion coeffi-
cients for the even expansion discussed above. Compute the max
error of \( u^{(m)}_N \) for \( m = 1, 2, 3, 4 \) and for \( N = 2^p, p = 2 - 10. \) Tabu-
late the results and discuss what you see, in particular in relation
to the results in the previous questions.

(d) Finally, use the differentiation matrix \( \mathcal{D} \) derived above to com-
pute \( u^{(m)}_N \) for \( m = 1, 2, 3, 4 \) for \( N = 2^p, p = 2 - 10 \) and compare
with the results in the two previous cases.

(e) Discuss any substantial differences between the 3 cases and try
to explain the differences and their sources, if you observe any
substantial differences.