AM256 - Homework 3

Due on Thursday, Feb 21, 2008

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1. Consider the discrete expansion

$$u_N(x) = \sum_{|n| \le N/2} \tilde{u}_n \exp(inx), \quad \tilde{u}_n = \frac{1}{c_n N} \sum_{j=0}^{N} u(x_j) \exp(-inx_j),$$

$$x_j = \frac{2\pi}{N} j, \quad j = 0, \dots, N-1,$$

with $c_n = 1$ for $|n| < N/2$ and $c_n = 2$ for $|n| = N/2$.

(a) Prove that one can express this as

$$u_N(x) = \sum_{j=0}^{N-1} u(x_j) g_j(x),$$

derive an explicit expression for $g_j(x)$ and show that it is a Lagrange polynomial.

(b) The derivative of $u_N(x)$ at the grid points can be expressed as

$$\frac{d}{dx} u_N(x_i) = \sum_{j=0}^{N-1} \mathcal{D}_{ij} u(x_j).$$

Derive the explicit entries of the differentiation matrix, $\mathcal{D}_{ij}$, and show that this matrix is a skew-symmetric, circulant, Toeplitz matrix.

(c) Show that $\mathcal{D}$ must have at least one zero eigenvalue (HINT: Consider a constant function).

2. Consider the following approximation to the integral

$$\frac{1}{2\pi} \int_0^{2\pi} u(x)\, dx \simeq \frac{1}{N+1} \sum_{j=0}^{N} u(x_j), \quad x_j = \frac{2\pi}{N+1} j, \quad j = 0, \dots, N.$$

(a) If we define the space of trigonometric functions as

$$\mathsf{B}_N = \text{span}\left\{\exp(inx)\right\}_{|n| \le N/2},$$

prove that the approximation is exact for all $u(x) \in \mathsf{B}_{2N}$.

(b) Confirm the analytic result by computations by choosing $u(x)$ to illustrate the accuracy and failure of the approximation

3. Consider the following function

$$u(x) = \frac{3}{5 - 4\cos(x)}, \quad x \in [0, 2\pi].$$

We shall explore ways of computing the derivatives of $u(x)$ using the continuous and discrete expansion coefficients as well as the differentiation matrix.

(a) Compute the analytic derivatives of $u(x)$ of up to 4th order, i.e., $u^{(m)}, m = 1, 2, 3, 4$.

(b) Using the exact continuous expansion coefficients

$$\hat{u}_n = 2^{-|n|},$$

compute the max error of $u_N^{(m)}$ for $m = 1, 2, 3, 4$ and for $N = 2^p, p = 2 - 10$. Tabulate the results and discuss what you see.

(c) Now repeat this exercise but use the discrete expansion coefficients for the even expansion discussed above. Compute the max error of $u_N^{(m)}$ for $m = 1, 2, 3, 4$ and for $N = 2^p, p = 2 - 10$. Tabulate the results and discuss what you see, in particular in relation to the results in the previous questions.

(d) Finally, use the differentiation matrix $\mathcal{D}$ derived above to compute $u_N^{(m)}$ for $m = 1, 2, 3, 4$ for $N = 2^p, p = 2 - 10$ and compare with the results in the two previous cases.

(e) Discuss any substantial differences between the 3 cases and try to explain the differences and their sources, if you observe any substantial differences.