APMA 2560 Homework #2 Solutions

February 15, 2008

1.) We have shown in class that if the operator $\mathcal{L}$ and the projection $\mathcal{P}_N$ commute as

$$\mathcal{P}_N \mathcal{L} \mathcal{P}_N u(x) = \mathcal{P}_N \mathcal{L}\, u(x),$$

then there is no truncation error and one can solve the problem exactly as long as the initial conditions are exactly represented within B_N.

For the Fourier expansion

$$\mathcal{P}_N u(x) = \sum_{|n| \leq N/2} \hat{u}_n\, e^{inx}, \qquad \hat{u}_n = \frac{1}{2\pi} \int_0^{2\pi} u(x)\, e^{-inx}\, \mathrm{d}x,$$

and $u(x)$ being periodic on $x \in [0, 2\pi]$.

a) What is B_N in this case?

b) Prove that if the operator is

$$\mathcal{L} = \frac{\mathrm{d}^q}{\mathrm{d}x^q},$$

the truncation error vanishes.

c) Prove that if the operator is

$$\mathcal{L} = \sin(x) \frac{\mathrm{d}}{\mathrm{d}x},$$

the truncation error does not vanish.

Solution:

a.) In the statement of the problem, we assume that N is even. In this case, the space B_N is given by,

$$B_N \;=\; \mathrm{span}\big\{e^{inx} : |n| \leq N/2\big\},$$

where the linear span is taken over the field of complex numbers.

b.) We take

$$\mathcal{L} = \frac{\mathrm{d}^q}{\mathrm{d}x^q},$$

and assume $u(x)$ any $L^2([0, 2\pi])$ function. Then $u(x)$ admits a Fourier expansion:

$$u(x) = \sum_{|n| < \infty} \hat{u}_n\, e^{inx}. \tag{1}$$

Given this expansion, then we have

$$\mathcal{P}_N u(x) = \sum_{|n| \le N/2} \hat{u}_n e^{inx}. \tag{2}$$

Of course, this implies that for all $u \in B_N$, we have $\mathcal{P}_N u = u$. We also have that for any $u \in L^2$,

$$\mathcal{L} u(x) = \sum_{|n| < \infty} (i\, n)^q \hat{u}_n e^{inx}.$$

Armed with these two statements, we can explicitly compute

$$\begin{aligned} \mathcal{P}_N \mathcal{L} \mathcal{P}_N u(x) &= \mathcal{P}_N \mathcal{L} \sum_{|n| \le N/2} \hat{u}_n e^{inx} \\ &= \mathcal{P}_N \sum_{|n| \le N/2} (i\, n)^q \hat{u}_n e^{inx} \\ &= \sum_{|n| \le N/2} (i\, n)^q \hat{u}_n e^{inx}. \end{aligned}$$

We also compute

$$\begin{aligned} \mathcal{P}_N \mathcal{L} u(x) &= \mathcal{P}_N \sum_{|n| < \infty} (i\, n)^q \hat{u}_n e^{inx} \\ &= \sum_{|n| \le N/2} (i\, n)^q \hat{u}_n e^{inx} \end{aligned}$$

We can now explicitly see that

$$\mathcal{P}_N \mathcal{L} \mathcal{P}_N u(x) = \mathcal{P}_N \mathcal{L} u(x),$$

which shows that the truncation error is zero. In this case, the space B_N is closed under the operator $\mathcal{L}$.

c.) We now take

$$\mathcal{L} = \sin(x) \frac{\mathrm{d}}{\mathrm{d}x}.$$

We still assume an expansion of u as given in (1) and the projection result (2) also still holds. For any $u \in L^2([0, 2\pi])$, we can compute

$$\begin{aligned} \mathcal{L} u(x) &= \sin(x) \sum_{|n| < \infty} (i\, n) \hat{u}_n e^{inx} \\ &= \sum_{|n| < \infty} \frac{n}{2} \hat{u}_n \left(e^{ix} - e^{-ix}\right) e^{inx} \\ &= \sum_{|n| < \infty} \left(\frac{n-1}{2} \hat{u}_{n-1} - \frac{n+1}{2} \hat{u}_{n+1}\right) e^{inx}. \end{aligned}$$

Note that this already shows that the space B_N is not closed under the operator $\mathcal{L}$. In any case, we compute

$$\begin{aligned}
\mathcal{P}_N \mathcal{L}\, \mathcal{P}_N u(x) &= \mathcal{P}_N \mathcal{L} \sum_{|n|\le N/2} \hat{u}_n\, e^{inx} \\
&= \mathcal{P}_N \Bigg[\sum_{|n|\le N/2-1} \left(\frac{n-1}{2}\, \hat{u}_{n-1} - \frac{n+1}{2}\, \hat{u}_{n+1} \right) e^{inx} \\
&\quad + \frac{N/2-1}{2}\, \hat{u}_{N/2-1}\, e^{ix\mathrm{N}/2} - \frac{-N/2+1}{2}\, \hat{u}_{-N/2+1}\, e^{-ixN/2} \\
&\quad + \frac{N/2}{2}\, \hat{u}_{N/2}\, e^{ix(N+1)/2} - \frac{-N/2}{2}\, \hat{u}_{-N/2}\, e^{-ix(N+1)/2} \Bigg] \\
&= \sum_{|n|\le N/2-1} \left(\frac{n-1}{2}\, \hat{u}_{n-1} - \frac{n+1}{2}\, \hat{u}_{n+1} \right) e^{inx} \\
&\quad + \frac{N/2-1}{2}\, \hat{u}_{N/2-1}\, e^{ix\mathrm{N}/2} - \frac{-N/2+1}{2}\, \hat{u}_{-N/2+1}\, e^{-ixN/2}.
\end{aligned}$$

We also have

$$\begin{aligned}
\mathcal{P}_N \mathcal{L}\, u(x) &= \mathcal{P}_N \sum_{|n|<\infty} \left(\frac{n-1}{2}\, \hat{u}_{n-1} - \frac{n+1}{2}\, \hat{u}_{n+1} \right) e^{inx} \\
&= \sum_{|n|\le N/2} \left(\frac{n-1}{2}\, \hat{u}_{n-1} - \frac{n+1}{2}\, \hat{u}_{n+1} \right) e^{inx}.
\end{aligned}$$

Comparing these last two derived expressions, we see that $\mathcal{P}_N\ \mathcal{L}\ \mathcal{P}_N\ u(x)$ is "missing" two terms: a $\hat{u}_{N/2+1}\, e^{ix\mathrm{N}/2}$ and a $\hat{u}_{-N/2-1}\, e^{-ixN/2}$. Thus, in this case the truncation error

$$\mathcal{P}_N \mathcal{L}(I - \mathcal{P}_N) u(x)$$

is not identically zero for any u. Note that if we assume that $u \in B_{N-1}$, however, then the truncation error is zero.

2.) For the linear wave equation

$$\frac{\partial u}{\partial t} = -c\, \frac{\partial u}{\partial x}, \qquad x \in [0, 2\pi]$$

we have considered ways of approximating the derivative using a Fourier series. Using the orthogonality of the basis, we have recovered that

$$\frac{\mathrm{d}\hat{u}_n}{\mathrm{d}t} = -i\, c\, n\, \hat{u}_n, \qquad |n| \le N/2. \tag{3}$$

a) Write this on the generic form

$$\frac{\mathrm{d}\boldsymbol{u}_N}{\mathrm{d}t} = \mathcal{L}_N \boldsymbol{u}_N,$$

and define what $\boldsymbol{u}_N$ and $\mathcal{L}_N$ mean.

b) Prove that the scheme is stable in the sense of

$$\|\exp(\mathcal{L}_N t)\| \le C(t),$$

and specify what $C(t)$ is.

Solution:

a.) We define the modal coefficients $\hat{u}_n$ to be

$$\hat{u}_n = \frac{1}{2\pi} \int_0^{2\pi} u(x)\, e^{-inx}\, \mathrm{d}x.$$

For a given N even, we arrange them in a vector as follows:

$$\boldsymbol{u}_N = \begin{pmatrix} \hat{u}_{-N/2} \\ \hat{u}_{-N/2+1} \\ \vdots \\ \hat{u}_{N/2-1} \\ \hat{u}_{N/2} \end{pmatrix}.$$

We define $\mathcal{L}_N$ to be a diagonal $(N+1) \times (N+1)$ matrix:

$$\mathcal{L}_N = \begin{pmatrix} \frac{i\,c\,N}{2} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \frac{i\,c\,(N-1)}{2} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \frac{i\,c\,(N-2)}{2} & & & \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & -\frac{i\,c\,(N-1)}{2} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & -\frac{i\,c\,N}{2} \end{pmatrix}$$

Then the semi-discrete form (3) can be written as

$$\frac{\mathrm{d}\boldsymbol{u}_N}{\mathrm{d}t} = \mathcal{L}_N \boldsymbol{u}_N.$$

b.) For a diagonalizable matrix $A = S\,\Lambda\,S^{-1}$, one way of defining the matrix exponential is

$$\exp(A) = S \exp(\Lambda)\, S^{-1},$$

where $\exp(\Lambda)$, the exponential of a diagonal matrix is simply the element-by-element exponential of the diagonal entries, and zero elsewhere. Since the operator $\mathcal{L}_N$ is already diagonal, we can directly apply this. For the norm, the 2-norm of a matrix is defined as

$$\|A\| = \sup_{x \neq 0} \frac{\|A\,x\|}{\|x\|},$$

where the norms on the elements x and $A\,x$ in this case can be taken as Euclidean 2-norms. One can then show from this that for a diagonalizable matrix A with eigenvalues λ_i, the matrix 2-norm is the same as the spectral radius $\rho(A)$:

$$\|A\| = \rho(A) = \max_i |\lambda_i|.$$

In other words, we have that

$$\|\exp(\Lambda\,t)\| = \max_{|n| \leq N/2} \left|e^{-icnt}\right| = 1.$$

So for the ODE (3), we have that

$$\|\exp(\Lambda\,t)\| = C(t) \equiv 1.$$

3.) Using the Fourier differentiation matrix, compute the derivative of the following functions

(a) $f(x) = \exp(\cos(4x))$

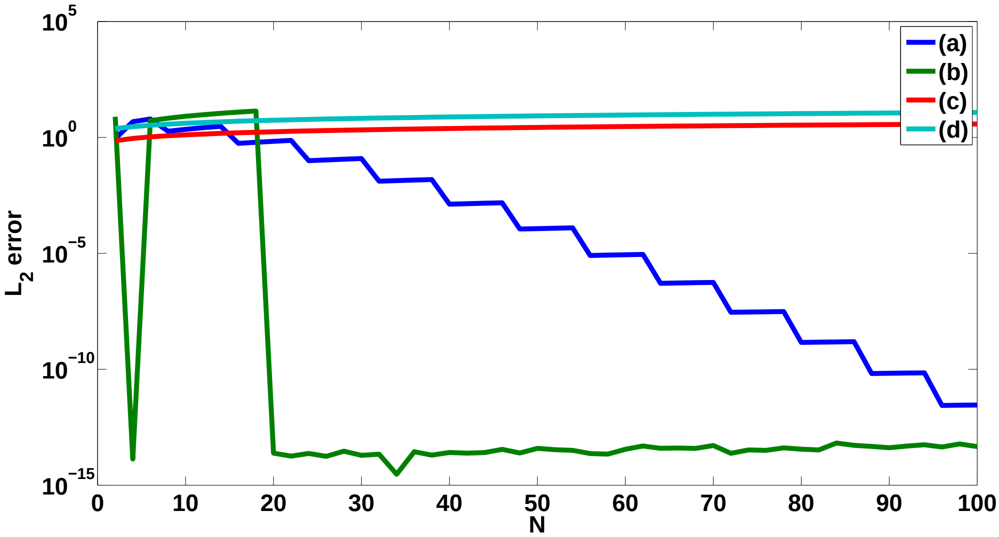
(b) $f(x) = \cos(10x)$
(c) $f(x) = \cos(x/2)$
(d) $f(x) = x$

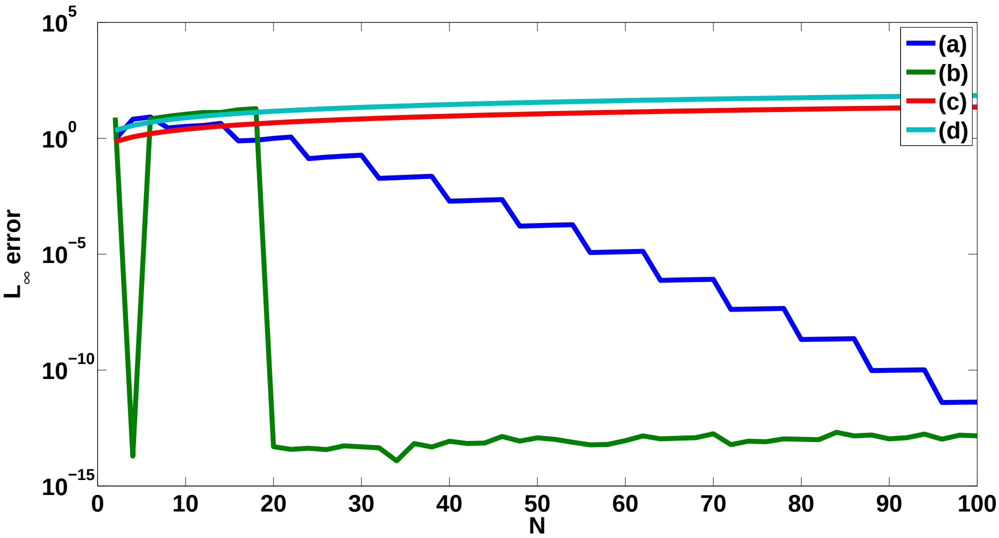
All functions are defined on $[0,2\pi]$.

Compute the pointwise error (L_∞) and the global error (L_2), for increasing values of N and discuss the different behaviors and convergence rates based on what you know from a theoretical point of view.

Solution:

The figures below were generated using MatLab's `semilogy` plotting utility.





(a) First note that $f(x)$ and all its derivatives are smoothly periodic on $[0,\ 2\pi]$, and so we expect spectral convergence in each norm. The analytical derivative is

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -4\sin(4\,x)\exp(\cos(4\,x))$$

This can be rewritten as

$$-\frac{4}{2i}\left(e^{4ix}-e^{-4ix}\right)\sum_{n}\left(e^{4ix}+e^{-4ix}\right)^{n}\frac{1}{n!\,2^{n}}$$

Note that this expansion contains only wavenumbers $4n$, where n is an integer. Numerically we observe spectral convergence in both norms, with the error decreasing as we are able to resolve each of the $4n$ wavenumbers. This corresponds to every 8 increments of N.

(b) The function $f(x)$ and all its first derivative are smoothly periodic with wavenumber 10. Therefore the function is only resolved for $N/2 \geq 10 \Rightarrow N \geq 20$. Numerically we see this as the error instantly hits machine precision for $N = 20$, completely resolving the wave.

(c) and (d) : These functions are not periodic and therefore we do not observe convergence in either L_2 or L_∞