1. We have shown in class that if the operator $L$ and the projection $P_N$ commute as

$$P_N L P_N u(x) = P_N L u(x),$$

then there is no truncation error and one can solve the problem exactly as long as the initial conditions are exactly represented within $B_N$.

For the Fourier expansion

$$P_N u(x) = \sum_{|n| \leq N/2} \hat{u}_n e^{inx}, \quad \hat{u}_n = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-inx} dx,$$

and $u(x)$ being periodic on $x \in [0, 2\pi]$.

a) What is $B_N$ in this case?

b) Prove that if the operator is

$$L = \frac{d^q}{dx^q},$$

the truncation error vanishes.

c) Prove that if the operator is

$$L = \sin(x) \frac{d}{dx},$$

the truncation error does not vanish.

Solution:

a.) In the statement of the problem, we assume that $N$ is even. In this case, the space $B_N$ is given by,

$$B_N = \text{span}\{ e^{inx} : |n| \leq N/2 \},$$

where the linear span is taken over the field of complex numbers.

b.) We take

$$L = \frac{d^q}{dx^q},$$

and assume $u(x)$ any $L^2([0, 2\pi])$ function. Then $u(x)$ admits a Fourier expansion:

$$u(x) = \sum_{|n| < \infty} \hat{u}_n e^{inx}. \quad (1)$$
Given this expansion, then we have

$$P_N u(x) = \sum_{|n| \leq N/2} \hat{u}_n e^{inx}. \quad (2)$$

Of course, this implies that for all $u \in B_N$, we have $P_N u = u$. We also have that for any $u \in L^2$,

$$L u(x) = \sum_{|n| < \infty} (i n)^q \hat{u}_n e^{inx}.$$

Armed with these two statements, we can explicitly compute

$$P_N L P_N u(x) = P_N L \sum_{|n| \leq N/2} \hat{u}_n e^{inx}$$

$$= P_N \sum_{|n| \leq N/2} (i n)^q \hat{u}_n e^{inx}$$

$$= \sum_{|n| \leq N/2} (i n)^q \hat{u}_n e^{inx}.$$

We also compute

$$P_N L u(x) = P_N \sum_{|n| < \infty} (i n)^q \hat{u}_n e^{inx}$$

$$= \sum_{|n| \leq N/2} (i n)^q \hat{u}_n e^{inx}.$$

We can now explicitly see that

$$P_N L P_N u(x) = P_N L u(x),$$

which shows that the truncation error is zero. In this case, the space $B_N$ is closed under the operator $L$.

c.) We now take

$$L = \sin(x) \frac{d}{dx}.$$

We still assume an expansion of $u$ as given in (1) and the projection result (2) also still holds. For any $u \in L^2([0, 2\pi])$, we can compute

$$L u(x) = \sin(x) \sum_{|n| < \infty} (i n) \hat{u}_n e^{inx}$$

$$= \sum_{|n| < \infty} \frac{n}{2} \hat{u}_n \left( e^{ix} - e^{-ix} \right) e^{inx}$$

$$= \sum_{|n| < \infty} \left( \frac{n-1}{2} \hat{u}_{n-1} - \frac{n+1}{2} \hat{u}_{n+1} \right) e^{inx}.$$ 

Note that this already shows that the space $B_N$ is not closed under the operator $L$. In any case,
we compute
\[ \mathcal{P}_N \mathcal{L} \mathcal{P}_N u(x) = \mathcal{P}_N \mathcal{L} \sum_{|n| \leq N/2} \hat{u}_n e^{inx} \]

\[ = \mathcal{P}_N \left[ \sum_{|n| \leq N/2-1} \left( \frac{n-1}{2} \hat{u}_{n-1} - \frac{n+1}{2} \hat{u}_{n+1} \right) e^{inx} + \frac{N/2-1}{2} \hat{u}_{N/2-1} e^{ixN/2} - \frac{N/2+1}{2} \hat{u}_{-N/2+1} e^{-ixN/2} \right] \]

\[ + \frac{N/2}{2} \hat{u}_{N/2} e^{ix(N+1)/2} - \frac{N/2}{2} \hat{u}_{-N/2} e^{-ix(N+1)/2} \]

\[ = \sum_{|n| \leq N/2-1} \left( \frac{n-1}{2} \hat{u}_{n-1} - \frac{n+1}{2} \hat{u}_{n+1} \right) e^{inx} + \frac{N/2-1}{2} \hat{u}_{N/2-1} e^{ixN/2} - \frac{N/2+1}{2} \hat{u}_{-N/2+1} e^{-ixN/2}. \]

We also have
\[ \mathcal{P}_N \mathcal{L} u(x) = \mathcal{P}_N \sum_{|n| < \infty} \left( \frac{n-1}{2} \hat{u}_{n-1} - \frac{n+1}{2} \hat{u}_{n+1} \right) e^{inx} \]

\[ = \sum_{|n| \leq N/2} \left( \frac{n-1}{2} \hat{u}_{n-1} - \frac{n+1}{2} \hat{u}_{n+1} \right) e^{inx}. \]

Comparing these last two derived expressions, we see that \( \mathcal{P}_N \mathcal{L} \mathcal{P}_N u(x) \) is “missing” two terms: a \( \hat{u}_{N/2+1} e^{ixN/2} \) and a \( \hat{u}_{-N/2-1} e^{-ixN/2} \). Thus, in this case the truncation error
\[ \mathcal{P}_N \mathcal{L} (I - \mathcal{P}_N) u(x) \]

is not identically zero for any \( u \). Note that if we assume that \( u \in B_{N-1} \), however, then the truncation error is zero.

2.) For the linear wave equation
\[ \frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x}, \quad x \in [0, 2\pi] \]

we have considered ways of approximating the derivative using a Fourier series. Using the orthogonality of the basis, we have recovered that
\[ \frac{\partial \hat{u}_n}{\partial t} = -icn \hat{u}_n, \quad |n| \leq N/2. \quad (3) \]

a) Write this on the generic form
\[ \frac{d\mathbf{u}_N}{dt} = \mathbf{L}_N \mathbf{u}_N, \]

and define what \( \mathbf{u}_N \) and \( \mathbf{L}_N \) mean.

b) Prove that the scheme is stable in the sense of
\[ \| \exp(\mathbf{L}_N t) \| \leq C(t), \]

and specify what \( C(t) \) is.
Solution:

a.) We define the modal coefficients $\hat{u}_n$ to be

$$\hat{u}_n = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-inx} \, dx.$$  

For a given $N$ even, we arrange them in a vector as follows:

$$u_N = \begin{pmatrix} \hat{u}_{-N/2} \\ \hat{u}_{-N/2+1} \\ \vdots \\ \hat{u}_{N/2-1} \\ \hat{u}_{N/2} \end{pmatrix}.$$  

We define $L_N$ to be a diagonal $(N+1) \times (N+1)$ matrix:

$$L_N = \begin{pmatrix} \frac{i\epsilon N}{2} & 0 & 0 & \ldots & 0 \\ 0 & \frac{i\epsilon (N-1)}{2} & 0 & \ldots & 0 \\ 0 & 0 & \frac{i\epsilon (N-2)}{2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & \frac{-i\epsilon (N-1)}{2} \end{pmatrix}.$$  

Then the semi-discrete form (3) can be written as

$$\frac{du_N}{dt} = L_N u_N.$$  

b.) For a diagonalizable matrix $A = S \Lambda S^{-1}$, one way of defining the matrix exponential is

$$\exp(A) = S \exp(\Lambda) S^{-1},$$  

where $\exp(\Lambda)$, the exponential of a diagonal matrix is simply the element-by-element exponential of the diagonal entries, and zero everywhere. Since the operator $L_N$ is already diagonal, we can directly apply this. For the norm, the 2-norm of a matrix is defined as

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|},$$  

where the norms on the elements $x$ and $Ax$ in this case can be taken as Euclidean 2-norms. One can then show from this that for a diagonalizable matrix $A$ with eigenvalues $\lambda_i$, the matrix 2-norm is the same as the spectral radius $\rho(A)$:

$$\|A\| = \rho(A) = \max_i |\lambda_i|.$$  

In other words, we have that

$$\|\exp(\Lambda t)\| = \max_{|n| \leq N/2} |e^{-icnt}| = 1.$$  

So for the ODE (3), we have that

$$\|\exp(\Lambda t)\| = C(t) \equiv 1.$$  

3.) Using the Fourier differentiation matrix, compute the derivative of the following functions

(a) $f(x) = \exp(\cos(4x))$
(b) \( f(x) = \cos(10x) \)  
(c) \( f(x) = \cos(x/2) \)  
(d) \( f(x) = x \)

All functions are defined on \([0, 2\pi]\).

Compute the pointwise error \((L_\infty)\) and the global error \((L_2)\), for increasing values of \(N\) and discuss the different behaviors and convergence rates based on what you know from a theoretical point of view.

**Solution:**

The figures below were generated using MatLab’s `semilogy` plotting utility.

(a) First note that \( f(x) \) and all its derivatives are smoothly periodic on \([0, 2\pi]\), and so we expect spectral convergence in each norm. The analytical derivative is

\[
\frac{du}{dx} = -4\sin(4x)\exp(\cos(4x))
\]
This can be rewritten as
\[-\frac{4}{2i}(e^{4ix} - e^{-4ix}) \sum_n (e^{4ix} + e^{-4ix})^n \frac{1}{n!2^n}\]

Note that this expansion contains only wavenumbers $4n$, where $n$ is an integer. Numerically we observe spectral convergence in both norms, with the error decreasing as we are able to resolve each of the $4n$ wavenumbers. This corresponds to every 8 increments of $N$.

(b) The function $f(x)$ and all its first derivative are smoothly periodic with wavenumber 10. Therefore the function is only resolved for $N/2 \geq 10 \Rightarrow N \geq 20$. Numerically we see this as the error instantly hits machine precision for $N = 20$, completely resolving the wave.

(c) and (d) : These functions are not periodic and therefore we do not observe convergence in either $L_2$ or $L_\infty$. 