AM117 Midterm, Sample Solutions

Part 1.

a) Consider

$$f(x) = x^2 - 3x + 1 + e^{x-1},$$

with

$$f'(x) = 2x - 3 + e^{x-1}$$

and

$$f''(x) = 2 + e^{x-1}.$$

We verify $f(1) = 1 - 3 + 1 + 1 = 0$ and $f'(1) = 2 - 3 + 1 = 0$. $f''(1) = 2 + 1 \neq 0$ confirms that 1 is exactly a double root (and no more).

b) We define $e_n := x_n - p$, where p is the root we seek. Then

$$\begin{aligned} e_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} - p \\ &= e_n - \frac{f(x_n)}{f'(x_n)} \\ &= \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)}. \end{aligned}$$

At this point, we recall Taylor's theorem:

$$0 = f(p) = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \frac{e_n^2}{2} f''(x_n + \xi)$$

for some $\xi \in (0, e_n)$. We rewrite this as

$$\frac{e_n^2}{2} f''(\xi) = e_n f'(x_n) - f(x_n),$$

which we plug into our expression above, to find

$$e_{n+1} = \frac{f''(\xi)}{2 f'(x_n)} e_n^2$$

Thus in general,

$$C = \frac{f''(\xi)}{2 f'(x_n)}, \quad \alpha = 2.$$

However if $f'(p) = 0$, applying the mean value theorem gives us

$$f''(\eta) = \frac{f'(x_n) - f'(p)}{x_n - p} = \frac{f'(x_n)}{e_n},$$

for some $\eta \in (x_n, p)$, or rather $f'(x_n) = f''(\eta) e_n$. Plugging this into our expression gives

$$e_{n+1} = \frac{f''(\xi)}{2 f''(\eta) e_n} e_n^2 = \frac{f''(\xi)}{2 f''(\eta)} e_n.$$

As $x_n \to p$, ξ and η to towards p also, and we obtain

$$C = \frac{1}{2}, \quad \alpha = 1.$$

[Thanks to Lo-Bin Chang for a nice solution.]

c) Either

$$g_1(x) = x - 2\frac{f(x)}{f'(x)}$$

or

$$g_2(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$

will restore second-order convergence. Since we know we're looking for a double root, we could also simply look for a root of f', provided f'' is available (which is also necessary for the second method).

Part 2.

We have

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 5 \end{pmatrix}.$$

NOTE: There's a typo in the eigenvalues in the problem. $\lambda_1(A)$ should be 0.6277, not 0.06277. This changes a few results further down. I'll accept results based on the wrong eigenvalue as right.

a) A theorem on p.214 of the book states this. Here's a proof, in case you're curious:

We need to show that $\boldsymbol{x}^T A \boldsymbol{x} > 0$ if $\boldsymbol{x} \neq \mathbf{0}$. Since A has three distinct eigenvalues and is a 3×3-matrix, there exist corresponding eigenvectors $\{\boldsymbol{v}_i\}_{i=1}^3$ that form an orthonormal basis of $\mathbb{R}^3$. (Why? Suppose $\boldsymbol{v}_i \neq \boldsymbol{v}_j$. Then

$$\begin{aligned} (1): \quad \boldsymbol{v}_i^T A \boldsymbol{v}_j &= \boldsymbol{v}_i^T \lambda_j \boldsymbol{v}_j \\ (2): \quad \boldsymbol{v}_i^T A \boldsymbol{v}_j &= (A\boldsymbol{v}_i)^T \boldsymbol{v}_j = \lambda_i \boldsymbol{v}_i^T \boldsymbol{v}_j \quad \text{(Symmetry!)} \\ (1-2): \quad 0 &= (\lambda_j - \lambda_i) \boldsymbol{v}_i^T \boldsymbol{v}_j. \end{aligned}$$

Since the eigenvalues are distinct, we have $(\lambda_j - \lambda_i) \neq 0$, so $\boldsymbol{v}_i^T \boldsymbol{v}_j = 0$, showing orthogonality. Since $\boldsymbol{v}_i \neq 0$ by the definition of an eigenvector, normalization yields an orthonormal basis.)

We may therefore write any vector $\boldsymbol{x} \in \mathbb{R}^3$ as $\boldsymbol{x} = \sum_i \alpha_i \boldsymbol{v}_i$. We find

$$\begin{aligned} \boldsymbol{x}^T A \boldsymbol{x} &= \left(\sum_{i=1}^3 \alpha_i \boldsymbol{v}_i \right)^T A \left(\sum_{j=1}^3 \alpha_j \boldsymbol{v}_j \right) \\ &= \left(\sum_{i=1}^3 \alpha_i \boldsymbol{v}_i \right)^T \left(\sum_{j=1}^3 \lambda_j \alpha_j \boldsymbol{v}_j \right) \\ &= \sum_{i,j=1}^3 \alpha_i \alpha_j \lambda_j \boldsymbol{v}_i^T \boldsymbol{v}_j = \sum_{i,j=1}^3 \alpha_i \alpha_j \lambda_j \delta_{i,j} \\ &= \sum_i \alpha_i^2 \lambda_j. \end{aligned}$$

Recall that the meaning of the so-called *Kronecker symbol* $\delta_{i,j}$ is

$$\delta_{i,j} := \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

By assumption, $\alpha_i^2 \lambda_j \geqslant 0$–all given eigenvalues are positive. Now if $\boldsymbol{x} \neq \boldsymbol{0}$, the α_i can't all have been zero, for some i, $\alpha_i^2 \lambda_j > 0$. Thus $\boldsymbol{x}^T A \boldsymbol{x} > 0$, showing the claim. □

b) In order to perform a Cholesky decomposition using Gaussian elimination, we recall how LU can be performed using Gaussian elimination, that is, by recording row operations in a lower diagonal matrix.

$$\begin{aligned}
A &= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 1 & & \\ \frac{1}{5} & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 1 \\ \langle 0 \rangle & \frac{4}{5} & \frac{4}{5} \\ 1 & 1 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 1 & & \\ \frac{1}{5} & 1 & \\ \frac{1}{5} & & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 1 \\ & \frac{4}{5} & \frac{4}{5} \\ \langle 0 \rangle & \frac{4}{5} & \frac{24}{5} \end{pmatrix} \\
&= \begin{pmatrix} 1 & & \\ \frac{1}{5} & 1 & \\ \frac{1}{5} & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 1 \\ & \frac{4}{5} & \frac{4}{5} \\ & \langle 0 \rangle & 4 \end{pmatrix}
\end{aligned}$$

We arrive at a valid LU decomposition. However, we are interested in $L\,L^T$, i.e. $U = L^T$. To achieve that, we pull the non-1 diagonal entries out of the right-hand matrix and split them using the square root:

$$\begin{aligned}
A &= \begin{pmatrix} 1 & & \\ \frac{1}{5} & 1 & \\ \frac{1}{5} & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & & \\ & \frac{4}{5} & \\ & & 4 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{5} & \frac{1}{5} \\ & 1 & 1 \\ & & 1 \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} 1 & & \\ \frac{1}{5} & 1 & \\ \frac{1}{5} & 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{5} & & \\ & \sqrt{\frac{4}{5}} & \\ & & 2 \end{pmatrix}}_{L:=} \begin{pmatrix} \sqrt{5} & & \\ & \sqrt{\frac{4}{5}} & \\ & & 2 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{5} & \frac{1}{5} \\ & 1 & 1 \\ & & 1 \end{pmatrix} \\
&= L\,L^T.
\end{aligned}$$

Observe how (predictably) after pulling out the diagonal matrix, the left and right matrices were already transposes of each other. We obtain

$$L = \begin{pmatrix} \sqrt{5} & & \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 2 \end{pmatrix}.$$

c) Pivoting strategies are applied to avoid small (or zero) pivots. For s.p.d. matrices, all pivots are guaranteed to be positive. Pivoting is therefore not needed.

Similarly, strictly diagonally dominant matrices do not need pivoting.

d) Inverting L gives

$$L^{-1} = \begin{pmatrix} \frac{1}{\sqrt{5}} & & \\ -\frac{\sqrt{5}}{10} & \frac{\sqrt{5}}{2} & \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then

$$A^{-1} = (L\,L^T)^{-1} = L^{-T} L^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -1 & \\ -1 & 6 & -1 \\ & -1 & 1 \end{pmatrix}.$$

e)

$$\begin{aligned} \rho(A) &= \max_i |\lambda_i(A)| = \lambda_3(A) \approx 6.3723. \\ \rho(A^{-1}) &= \max_i |\lambda_i(A^{-1})| = \max_i \frac{1}{|\lambda_i(A)|} = \frac{1}{\min_i |\lambda_i(A)|} = \frac{1}{|\lambda_1(A)|} \approx \frac{1}{0.6277} \approx 1.5931. \end{aligned}$$

With the wrong eigenvalue, you would have obtained

$$\rho(A^{-1}) \approx \frac{1}{0.06277} \approx 15.931.$$

f) We have $\kappa_1(A) = \kappa_\infty(A)$ because $A = A^T$ implies $\|A\|_1 = \|A\|_\infty$ and $(A^{-1})^T = A^{-1}$, so also $\|A^{-1}\|_1 = \|A^{-1}\|_\infty$.

$$\begin{aligned} \kappa_\infty(A) &= \|A\|_\infty \|A^{-1}\|_\infty = 7 \cdot 2 = 14. \\ \kappa_2(A) &= \|A\|_2 \|A^{-1}\|_2 = \rho(A)\,\rho(A^{-1}) \approx 10.15. \\ \kappa_1(A) &= \kappa_\infty(A) = 14. \end{aligned}$$

With the wrong eigenvalue, you would have obtained

$$\kappa_2(A) \approx 101.5.$$

There's no reason why $\kappa_2(A)$ should match the other two.

g) Recall the Neumann series and its value,

$$(\mathrm{Id} - T)^{-1} = \sum_{i=0}^{\infty} T.$$

To express A^{-1} with it, we would need $A = \mathrm{Id} - T$, i.e. $T = \mathrm{Id} - A$. Now consider

$$\begin{aligned} \lambda_1(T) = \lambda_1(\mathrm{Id} - A) &= 1 - \lambda_1(A) \approx 0.93723. \\ \lambda_2(T) = \lambda_2(\mathrm{Id} - A) &= 1 - \lambda_2(A) = -3. \\ \lambda_3(T) = \lambda_3(\mathrm{Id} - A) &= 1 - \lambda_3(A) \approx -5.3723. \end{aligned}$$

For the Neumann series to converge, we require $\rho(T) < 1$. That is obviously not the case.

Part 3.

Given

$$A = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}.$$

a) Quite obviously $A = A^T \Leftrightarrow a = b$.

b) A is diagonally dominant iff

$$|a_{ii}| >= \sum_{j=1, j \neq i}^{n} |a_{ij}| \quad \text{for} \quad i = 1, \ldots, n.$$

For our matrix, this boils down to $1 \geqslant |a|, \quad 1 \geqslant |b|$.

c)

$$\begin{aligned}
\begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} &= x(x + a\,y) + y(b\,x + y) \\
&= x^2 + (a+b)\,x\,y + y^2. \\
&= \left(x + \frac{a+b}{2} y\right)^2 - \frac{(a+b)^2}{4} y^2 + y^2 \\
&= \underbrace{\left(x + \frac{a+b}{2} y\right)^2}_{\geqslant 0} + \left(1 - \frac{(a+b)^2}{4}\right) y^2
\end{aligned}$$

A close look at this result reveals that we can always make the first term disappear by a suitable choice of x. Therefore,

$$\left(1 - \frac{(a+b)^2}{4}\right) > 0 \Leftrightarrow \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} > 0 \quad \text{for all } x, y.$$

The condition can be simplified a little:

$$\begin{aligned}
1 - \frac{(a+b)^2}{4} &> 0 \\
\Leftrightarrow (a+b)^2 &< 4. \\
\Leftrightarrow a + b &\in (-2, 2).
\end{aligned}$$

So A is positive definite iff $a + b \in (-2, 2)$.

Extra credit.

Let $A \in \mathbb{R}^{n \times n}$ be a matrix with $A = -A^T$, and let $\boldsymbol{x} \in \mathbb{R}^n$ be a vector.

$$\begin{aligned}
\boldsymbol{x}^T A \boldsymbol{x} &= (\boldsymbol{x}^T A \boldsymbol{x})^T \quad \text{(this is a scalar, so we may “transpose” it)} \\
&= \boldsymbol{x}^T A^T (\boldsymbol{x}^T)^T \quad \text{(reverse the order, remember?)} \\
&= \boldsymbol{x}^T A^T \boldsymbol{x} \quad ((\boldsymbol{x}^T)^T = \boldsymbol{x}) \\
&= -\boldsymbol{x}^T A \boldsymbol{x} \quad \text{(use skew-symmetry).}
\end{aligned}$$

In summary $\boldsymbol{x}^T A \boldsymbol{x} = -\boldsymbol{x}^T A \boldsymbol{x}$. But there's only one number for which that is true, and that is zero. Thus $\boldsymbol{x}^T A \boldsymbol{x} = 0$. □