APMA1170 Homework 13, Sample Solutions.

1. Problem 1, p. 318

Let $Q \in \mathbb{R}^{n\times n}$ be orthogonal, i.e. $Q^TQ = I$.

a) $(Q\boldsymbol{x})^TQ\boldsymbol{y} = \boldsymbol{x}^T\underbrace{Q^TQ}_{I}\boldsymbol{y} = \boldsymbol{x}^T\boldsymbol{y}$.

b) Suppose $Q\boldsymbol{x} = \lambda\boldsymbol{x}$. Then

$$\|Q\boldsymbol{x}\|_2^2 = (\lambda\boldsymbol{x})^T\lambda\boldsymbol{x} = \lambda^2\boldsymbol{x}^T\boldsymbol{x}.$$

On the other hand,

$$\|Q\boldsymbol{x}\|_2^2 = (Q\boldsymbol{x})^TQ\boldsymbol{x} \overset{a)}{=} \boldsymbol{x}^T\boldsymbol{x}.$$

Consequently, $|\lambda| = 1$.

c) Since all eigenvalues of Q have absolute value 1 (cf. b), the spectral radius (which is just the maximum absolute value of all eigenvalues) must be 1 as well, i.e. $\rho(Q) = 1$.

d) $\|Q\|_2 = \sqrt{\rho(Q^TQ)} = \sqrt{\rho(\mathrm{Id})} = 1$.

e) Since $Q^{-1} = Q^T$ is orthogonal just like Q (consider $(Q^T)^TQ^T = \mathrm{Id}$) it satisfies the assumption of part d), thus $\|Q^{-1}\| = 1$, and so $\kappa_2(Q) = \|Q\|_2\|Q^{-1}\|_2 = 1$.

2. Problems 8, 12, p. 319

Results:

```
--------------------------------------------
Problem 8
--------------------------------------------
reduced_A =

   4.05263   1.19092   0.00000  -0.00000
   1.19092   2.44737  -2.17945   0.00000
   0.00000  -2.17945   2.50000  -1.41421
  -0.00000   0.00000  -1.41421   5.00000

vectors =

   0.68825   0.16222  -0.70711   0.00000
   0.68825   0.16222   0.70711   0.00000
   0.22942  -0.97333  -0.00000   0.00000
   0.00000   0.00000   0.00000   1.00000

norm_V_T_Vt_minus_A =  2.0551e-15
--------------------------------------------
Problem 12
--------------------------------------------
reduced_A =

   1.6917e+01  -6.0532e-01   8.8818e-16   1.9780e-15   4.0940e-17
  -6.0532e-01   1.8282e+01  -5.3262e+00  -3.1816e-15  -5.1761e-16
   8.8818e-16  -5.3262e+00   2.2318e+01   1.2130e+01   9.5587e-16
   1.9780e-15  -2.9595e-15   1.2130e+01   1.4483e+01  -7.6158e+00
   4.0940e-17  -5.1761e-16   9.5587e-16  -7.6158e+00   1.2000e+01
```

```
vectors =

  -0.10901  -0.49731  -0.55657   0.65653   0.00000
  -0.76720  -0.31843  -0.18478  -0.52523   0.00000
  -0.63206   0.46764   0.32550   0.52523   0.00000
  -0.00446   0.65772  -0.74171  -0.13131   0.00000
   0.00000   0.00000   0.00000   0.00000   1.00000

norm_V_T_Vt_minus_A =  2.0237e-14
```

Code:

```
function p2
  disp('--------------------------------------------')
  disp('Problem 8')
  disp('--------------------------------------------')

  A = [...
  4,1,-2,1;...
  1,3,1,-1;...
  -2,1,2,0;...
  1,-1,0,5];

  [reduced_A, vectors] = reduce_sym_to_tri(A)

  norm_V_T_Vt_minus_A = norm(vectors*reduced_A*vectors' - A, 2)

  disp('--------------------------------------------')
  disp('Problem 12')
  disp('--------------------------------------------')

  A = [...
  6,2,-1,-3,-5;...
  2,18,-2,5,4;...
  -1,-2,20,-5,-4;...
  -3,5,-5,28,1;...
  -5,4,-4,1,12];

  [reduced_A, vectors] = reduce_sym_to_tri(A)

  norm_V_T_Vt_minus_A = norm(vectors*reduced_A*vectors' - A, 2)
end
% ---------------------------------------------------------------------------
function [T, V] = reduce_sym_to_tri(A)
  [n, dummy] = size(A);
  T = A;
  V = eye(n);

  for column = n:-1:3
    x = T(:,column);
    k = 1+n-column;

    xpart = x(1:n-k);
```

```
    mysign = 1;
    if (x(n-k)<0)
      mysign = -1;
    end
    alpha = -mysign*sqrt(xpart'*xpart);

    w = zeros(n,1);
    w(n-k) = sqrt(1/2*(1-x(n-k)/alpha));
    w(1:n-k-1) = -1/2 * x(1:n-k-1)/(alpha*w(n-k));

    u = T*w;
    K = w'*u;
    q = u - K*w;
    T = T - 2*w*q' -2*q*w';

    V = V - 2*(V*w)*w';
  end

  assert(abs(T - V'*A*V) < 1e-14)
end
% ----------------------------------------------------------------------------
function assert(p)
  not_p = ~p;
  if (sum(abs(not_p)) ~= 0)
    error('Assertion violated.')
  end
end
```